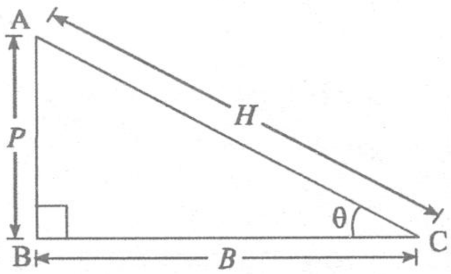
Trigonometry

Trigonometry:

Trigonometry means, the science which deals with the measurement of triangles.

(a) `Trigonometric Ratios :



A right angled triangle is shown in **Figure.** $\angle B$ Is of 90^0 Side opposite to $\angle B$ be called **hypotenuse.** There are two other angles i.e. $\angle A$ and $\angle C$. It we consider $\angle C$ as θ, then opposite side to this angle is called **perpendicular** and side adjacent to θ is called base.

(i) **Six Trigonometry Ratio are :**

$$\sin\theta = \frac{\text{Perpenicular}}{\text{Hypotenuse}} = \frac{P}{H} = \frac{AB}{AC} \qquad \cos\text{es}\theta = \frac{\text{Hypoteuse}}{\text{Perpendicular}} = \frac{H}{P} = \frac{AC}{AB}$$

$$\cos\theta = \frac{\text{Base}}{\text{Hypotenuse}} = \frac{B}{H} = \frac{BC}{AC} \qquad \sec\theta = \frac{\text{Hypotenuse}}{\text{Base}} = \frac{H}{B} = \frac{AC}{BC}$$

$$\tan\theta = \frac{\text{Perpendicular}}{\text{Base}} = \frac{P}{B} = \frac{AB}{BC} \qquad \cot\theta = \frac{\text{Base}}{\text{Parpendicular}} = \frac{B}{P} = \frac{BC}{AB}$$

(ii) Interrelationship is Basic Trigonometric Ratio :

$$\tan\theta = \frac{1}{\cot\theta} \Rightarrow \cot\theta = \frac{1}{\tan\theta}$$

$$\cos\theta = \frac{1}{\sec\theta} \Rightarrow \sec\theta = \frac{1}{\cos\theta}$$

$$\sin\theta = \frac{1}{\cos\text{ec}\theta} \Rightarrow \cos\text{ec}\theta = \frac{1}{\sin\theta}$$

We also observe that

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \Rightarrow \cot\theta = \frac{\cos\theta}{\sin\theta}$$

(b) Trigonometric Table :

θ →	0	30^0	45^0	60^0	90^0
Sin	**0**	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	**1**
Cos	**1**	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	**0**
Tan	**0**	$\frac{1}{\sqrt{3}}$	**1**	$\sqrt{3}$	**Not defined**
Cot	**Not defined**	$\sqrt{3}$	**1**	$\frac{1}{\sqrt{3}}$	**0**
Sec	**1**	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	**2**	**Not defined**
Cosec	**Not defined**	**2**	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	**1**

(c) Trigonometric Identities :

(i) $\sin^2\theta + \cos^2\theta = 1$

(A) $\sin^2\theta = 1 - \cos^2\theta$

(B) $\cos^2\theta = 1 - \sin 2\theta$

(ii) $1 + \tan^2\theta = \sec^2\theta$

(A) $\sec^2\theta - 1 = \tan^2\theta$

(B) $\sec^2\theta - \tan^2\theta = 1$

(C) $\tan^2\theta - \sec^2\theta = -1$

(iii) $1 + \cot^2\theta = \text{cos ec}^2\theta$

(A) $\text{cos ec}^2\theta - 1 = \cot^2\theta$

(B) $\text{cos ec}^2\theta - \cot^2\theta = 1$

(C) $\cot^2\theta - \text{cos ec}^2\theta = -1$

(d) Trigonometric Ratio of Complementary Angles:

$\sin(90-\theta) = \cos\theta$ $\quad$ $\cos(90-\theta) = \sin\theta$

$\tan(90-\theta) = \cot\theta$ $\quad$ $\cot(90-\theta) = \tan\theta$

$\sec(90-\theta) = \text{cos ec}\,\theta$ $\quad$ $\text{cos ec}(90-\theta) = \sec\theta$

Illustrations:

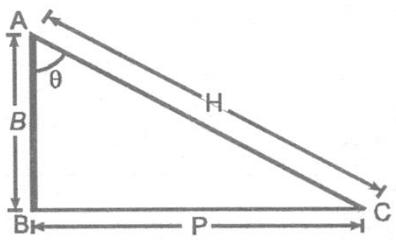
Example: In the given triangle AB = 3 cm and AC = 5 cm. Find all trigonometric ratios.

Sol. Using Pythagoras theorem

$AC^2 = AB^2 + BC^2$

$\Rightarrow$ $5^2 = 3^2 + p^2$

$\Rightarrow$ $16 = p^2$ $\quad\Rightarrow\quad$ P = cm



Here P = 4 cm, B = 3 cm, H = 5 cm

$\therefore \quad \sin\theta = \frac{P}{H} = \frac{4}{5}$

$\cos\theta = \frac{B}{H} = \frac{3}{5}$

$\tan\theta = \frac{P}{B} = \frac{4}{3}$

$\cot\theta = \frac{B}{P} = \frac{3}{4}$

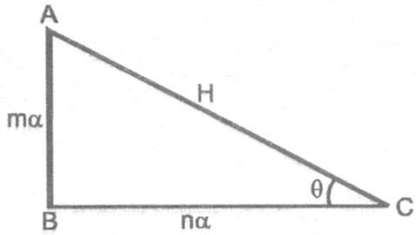
$\sec\theta = \frac{H}{B} = \frac{5}{3}$

$\operatorname{cosec}\theta = \frac{H}{P} = \frac{5}{4}$

Example: If $\tan\theta = \frac{m}{n}$, then find $\sin\theta$.

Sol. Let $P = m\alpha$ and $B = n\alpha$

$\therefore \quad \tan\theta = \frac{P}{B} = \frac{m}{n}$



$H^2 = P^2 + B^2$

$H^2 = m^2\alpha^2 + n^2\alpha^2$

$H = \alpha\sqrt{m^2 + n^2}$

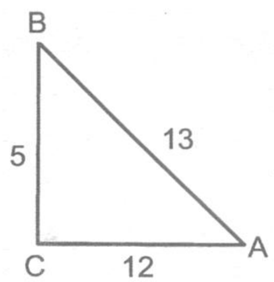
$\therefore \quad \tan\theta = \frac{P}{H} = \frac{ma}{a\sqrt{m^2 + n^2}}$

$\sin\theta = \frac{m}{\sqrt{m^2 + n^2}}$

Example: If cosec $A = \frac{13}{5}$ the prove than $\tan^2A$-sing$^2A = \sin^4A \sec^2 A$.

Sol. We hare coses A $= \frac{13}{5} = \frac{\text{Hypotenuse}}{\text{Perpendicular}}$

So, we draw a right triangle ABC, right angled at C such that hypotenuse AB = 13 units and perpendicular BC = 5 units



B Pythagoras theorem,

$AB^2 = BC^2 + AC^2 \Rightarrow (13)^2 = (5)^2 + AC^2$

$AC^2 = 169 - 25 = 144$

$AC = \sqrt{144} = 12$ units

$$\tan A = \frac{BC}{AC} = \frac{5}{12}$$

$$\sin A = \frac{BC}{AB} = \frac{5}{13}$$

and $$\sec A = \frac{AB}{AC} = \frac{13}{12}$$

L.H.S. $\tan^2 A - \sin^2 A$

$$= \left(\frac{5}{12}\right)^2 - \left(\frac{5}{13}\right)^2$$

$$= \frac{25}{144} - \frac{25}{169}$$

$$= \frac{25(169 - 144)}{144 \times 169}$$

$$= \frac{25 \times 25}{144 \times 169}$$

R.H.S. $= \sin^4 A \times \sec^2 A$

$$= \left(\frac{5}{13}\right)^4 \times \left(\frac{13}{12}\right)^2$$

$$= \frac{5^4 \times 13^2}{13^4 \times 12^2}$$

$$= \frac{5^4}{13^2 \times 12^2}$$

$$= \frac{25 \times 25}{144 \times 169}$$

So, L.H.S. = R.H.S. **Hence Proved.**

Example: In Δ ABC, right angled at B. AC + AB = 9 cm. Determine the value of cot C, cosec C, sec C.

Sol. In Δ ABC, we have

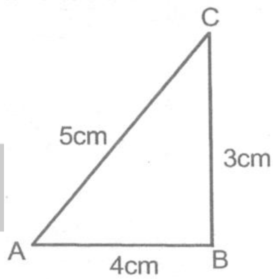
$(AC)^2 = (AB)^2 + BC^2$

$\Rightarrow$ $(9 - AB)^2 = AB^2 + (3)^2$ [$\because$ AC + AB = 9cm $\Rightarrow$ AC = 9 - AB]

$\Rightarrow$ $(81 + AB^2 - 18AB = AB^2 + 9$

$\Rightarrow$ $72 - 18\,AB = 0$

$\Rightarrow$ $AB = \frac{72}{18} = 4$ cm.



Now, AC + AB = 9 cm

AC = 9 - 4 = 5 cm

So, $\cot C = \frac{BC}{AB} = \frac{3}{4}$, $\text{cosec } C = \frac{AC}{AB} = -\frac{5}{4}$, $\sec C = \frac{AC}{BC} = \frac{5}{3}$.

Example: Given that cos (A - B) = cos A cos B + Sin B, find the value of cos 15^0.

Sol. Putting $A = 45^0$ and $B = 30^0$

We get $\cos(45^0 - 30^0) = \cos 45^0 \cos 30^0 + \sin 45^0 \sin 30^0$

$$\Rightarrow \cos 15^0 = \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2}$$

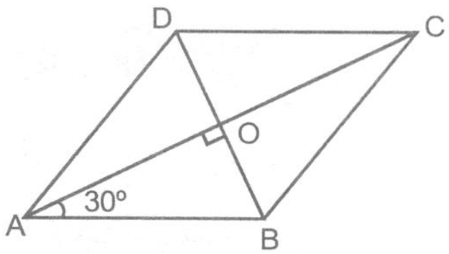
$$\Rightarrow \cos 15^0 = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

Example: A Rhombus of side of 10 cm has two angles of 60^0 each. Find the length of diagonals and also find its area.

Sol. Let ABCD be a rhombus of side 10 cm and $\angle BAD = \angle BCD = 60^0$. Diagonals of parallelogram bisect each other.

S, AO = OC and BO = OD

In right triangle AOB



$\sin 30^0 = \frac{OB}{AB}$ $\qquad$ $\cos 30^0 = \frac{OA}{AB}$

$\Rightarrow \frac{1}{2} = \frac{OB}{10}$ $\qquad$ $\Rightarrow \frac{\sqrt{3}}{2} = \frac{OA}{10}$

$\Rightarrow$ OB = 5 cm $\qquad$ $\Rightarrow$ $OA = 5\sqrt{3}$

$\therefore$ BD = 2 (OB) $\qquad$ $\Rightarrow$ AC = 2(OA)

$\Rightarrow$ BD = 2(5) $\qquad$ $\Rightarrow$ $AC = 2\,(5\sqrt{3})$

$\Rightarrow$ BD = 10 cm $\qquad$ $\Rightarrow$ $AC = 10\sqrt{3}$ cm

So, the length of diagonals AC = $10\sqrt{3}$ cm & BD = 10 cm

Area of Rhombus $= \frac{1}{2} \times AC \times BD$

$= \frac{1}{2} \times 10\sqrt{3} \times 10$

$= 50\sqrt{3}\text{cm}^2.$

Example: Evaluate : $\frac{\sec^2 54^0 - \cot^2 36^0}{\cos ec^2 57^0 - \tan^2 33^0} + 2\sin^2 38^0 \sec^2 52^0 - \sin^2 45^0 + \frac{2}{\sqrt{3}} \tan 17^0 \tan 60^0 \tan 73^0$

Sol. $\frac{\sec^2 54^0 - \cot^2 36^0}{\cos ec^2 57^0 - \tan^2 33^0} + 2\sin^2 38^0 \sec^2 52^0 - \sin^2 45^0 + \frac{2}{\sqrt{3}} \tan 17^0 \tan 60^0 \tan 73^0$

$= \frac{\sec^2(90^0 - 36^0) - \cot^2 36^0}{\cos ec^2(90^0 - 33^0) - \tan^2 33^0} + 2\sin^2 38^0 \sec^2(90^0 - 38^0) - \sin^2 45^0 + \frac{2}{\sqrt{3}} \tan(90^0 - 73^0) \tan 73^0 \tan 60^0$

$= \frac{\cos ec^2 36^0 - \cot^2 36^0}{\sec^2 33^0 - \tan^2 33^0} + 2\sin^2 38^0 \cos ec^2 38^0 - \left(\frac{1}{\sqrt{2}}\right)^2 + \frac{2}{\sqrt{3}} \cot 73^0 \tan 73^0 \times \sqrt{3}$

$= \frac{1}{1} + 2\sin^2 38^0 \times \frac{1}{\sin^2 38^0} - \frac{1}{2} + \frac{2}{\sqrt{3}} \times \frac{1}{\tan 73^0} \times 73^0 \times \sqrt{3} [\because \cos ec^2\theta - \cot^2\theta = 1, \sec^2\theta - \tan^2\theta = 1]$

$= 1 + 2 - \frac{1}{2} + 2 = 5 - \frac{1}{2}$

$= \frac{9}{2}.$

Example: Prove that : $\operatorname{cosec}(65^0+\theta)-\sec(25^0-\theta)-\tan(55^0-\theta)+\cot(35^0+\theta)=0$

Sol. $\operatorname{cosec}(65^0+\theta)=\operatorname{cosec}\{90^0-(25^0-\theta)\}=\sec(25^0-\theta)$ .....(i)

$\cot(35^0+\theta)=\cot\{90^0-(55^0-\theta)\}=\tan(55^0-\theta)$ .....(ii)

$\therefore$ **L.H.S.** $\operatorname{cosec}(65^0+\theta)-\sec(25^0-\theta)-\tan(55^0-\theta)+\cot(35^0+\theta)$

$=\sec(25^0-\theta)-\sec(25^0-\theta)-\tan(55^0-\theta)+\tan(55^0-\theta)$

$=0$ [using (i) & (ii)] **R.H.S.**

Example: Prove that : $\cot\theta-\tan\theta=\dfrac{2\cos^2\theta-1}{\sin\theta\cos\theta}$

Sol. **L.H.S.** $\cot\theta-\tan\theta$

$=\dfrac{\cos\theta}{\sin\theta}-\dfrac{\sin\theta}{\cos\theta}$ $\left[\because \cot\theta=\dfrac{\cos\theta}{\sin\theta}, \tan\theta=\dfrac{\sin\theta}{\cos\theta}\right]$

$=\dfrac{\cos^2\theta-\sin^2}{\sin\theta\cos\theta}$ $=\dfrac{\cos^2\theta-(1-\cos^2\theta)}{\sin\theta\cos\theta}$ $[\because \sin^2\theta=1-\cos^2\theta]$

$=\dfrac{\cos^2\theta-1+\cos^2\theta}{\sin\theta\cos\theta}$ $=\dfrac{2\cos^2\theta-1}{\sin\theta\cos\theta}$

R.H.S. **Hence Proved.**

Example: Prove that : (coses A – sin A) (sec A – cos A) (tan A + cot A) = 1.

Sol. L.H.S. (cosec A – sin A) (sec A – cos A) (tan A + cot A)

$=\left(\dfrac{1}{\sin A}-\sin A\right)\left(\dfrac{1}{\cos A}-\cos A\right)\left(\dfrac{\sin A}{\cos A}+\dfrac{\cos A}{\sin A}\right)$

$=\left(\dfrac{1-\sin^2 A}{\sin A}\right)\left(\dfrac{1-\cos^2 A}{\cos A}\right)\left(\dfrac{\sin^2 A+\cos^2 A}{\sin A\cos A}\right)$

$=\left(\dfrac{\cos^2 A}{\sin A}\right)\left(\dfrac{\sin^2 A}{\cos A}\right)\left(\dfrac{1}{\sin A\cos A}\right)$ $[\because \sin^2 A+\cos^2 A=1]$

$=1$ **R.H.S.** **Hence Proved.**

Example: If $\sin\theta+\cos\theta=m$ and $\sec\theta+\operatorname{cosec}\theta=n$, then prove that $n(m^2-1)=2m$.

Sol. **L.H.S.** $n(m^2-1)$

$=(\sec\theta+\operatorname{cosec}\theta)[(\sin\theta+\cos\theta)^2-1]$

$=\left(\dfrac{1}{\cos\theta}+\dfrac{1}{\sin\theta}\right)(\sin^2\theta+\cos^2\theta+2\sin\theta\cos\theta-1)$

$$= \left(\frac{\cos\theta + \sin\theta}{\sin\theta\cos\theta}\right)(1 + 2\sin\theta\cos\theta - 1)$$

$$= \frac{(\cos\theta + \sin\theta)}{\sin\theta\cos\theta}(2\sin\theta\cos\theta)$$

$$= 2(\sin\theta + \cos\theta)$$

$= 2m$ **R.H.S. Hence Proved.**

Example: If $\sec\theta = x + \frac{1}{4x}$, then prove that $\sec\theta + \tan\theta = 2x$ or $\frac{1}{2x}$.

Sol. $\sec\theta = x + \frac{1}{4x}$ ....(i)

$\therefore \quad 1 + \tan^2\theta = \sec^2\theta$

$\Rightarrow \quad \tan^2\theta = \sec^2\theta - 1 \quad \Rightarrow \quad \tan^2\theta = \left(x + \frac{1}{4x}\right)^2 - 1$

$\Rightarrow \quad \tan^2\theta = x^2 + \frac{1}{16x^2} + 2 \times x \times \frac{1}{4x} - 1$

$\Rightarrow \quad \tan^2\theta = x^2 + \frac{1}{16x^2} + \frac{1}{2} - 1 \quad \Rightarrow \quad \tan\theta = \pm\left(x - \frac{1}{4x}\right)$

$\Rightarrow \quad \tan^2\theta = \left(x - \frac{1}{4x}\right)^2 \quad \Rightarrow \quad \tan\theta = \pm\left(x - \frac{1}{4x}\right)$

So, $\tan\theta = x - \frac{1}{4x}$ ....(ii)

or $\tan\theta = -\left(x - \frac{1}{4x}\right)$ ....(iii)

Adding equation (i) and (ii)

$$\sec\theta + \tan\theta = x + \frac{1}{4x} + x - \frac{1}{4x}$$

$$\sec\theta + \tan\theta = 2x$$

Adding equation (i) and (ii)

$$\sec\theta + \tan\theta = x + \frac{1}{4x} - x + \frac{1}{4x}$$

$$= \frac{1}{2x}$$

Hence, $\sec\theta + \tan\theta + 2x$ or $\dfrac{1}{2x}$.

Example: If θ is an acute angle and $\tan\theta + \cot\theta = 2$ find the value of $\tan^9\theta + \cot^9\theta$

Sol. We have, $\tan\theta + \cot\theta = 2$

$\Rightarrow \quad \tan\theta + \dfrac{1}{\tan\theta} = 2$

$\Rightarrow \quad \dfrac{\tan^2\theta + 1}{\tan\theta} = 2$

$\Rightarrow \quad \tan^2\theta + 1 = 2\tan\theta$

$\Rightarrow \quad \tan^2\theta - 2\tan\theta + 1 = 0$

$\Rightarrow \quad (\tan\theta - 1)^2 = 0$

$\Rightarrow \quad \tan\theta - 1 = 0$

$\Rightarrow \quad \tan\theta = 1$

$\Rightarrow \quad \tan\theta = \tan 45^0$

$\Rightarrow \quad \theta = 45^0$

$\therefore \quad \tan^9\theta + \cot^9\theta$

$= \quad \tan^9 45^0 + \cot^9 45^0$

$= \quad (\tan 45)^9 + (\cot 45)^0$

$= \quad (1)^9 + (1)^9$

$= \quad 2.$