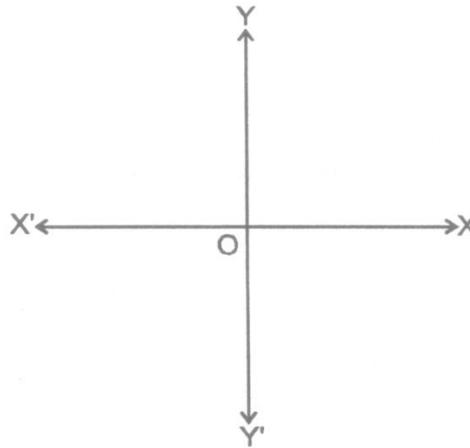# Co-ordinate Geometry

Rectangular Co-Ordinates:

Take two perpendicular lines **X'OX** and **Y'OY** intersecting at the point **O. X'OX** and **Y'OY** are called the co-ordinate axes. **X'Ox** is called the **X-axis, Y'OY** is called the **Y-axis** and **O** is called the origin. Lines **X'OX** and **Y'OY** are sometimes also called rectangular axes.

(a)   Co-ordinates of a Point :

Let **P** be any point as shown in figure. Draw **PL** and **PM** perpendiculars on **Y-axis** and **X-axis,** respectively. The length **LP (or OM)** is called the **x - coordinate of the abscissa of point P** and MP i called the **y - coordinate or the ordinate of point P.** A point whose abscissa is **x** and ordinate is **y** named as the point **(x,y)** or **P(x,y).**
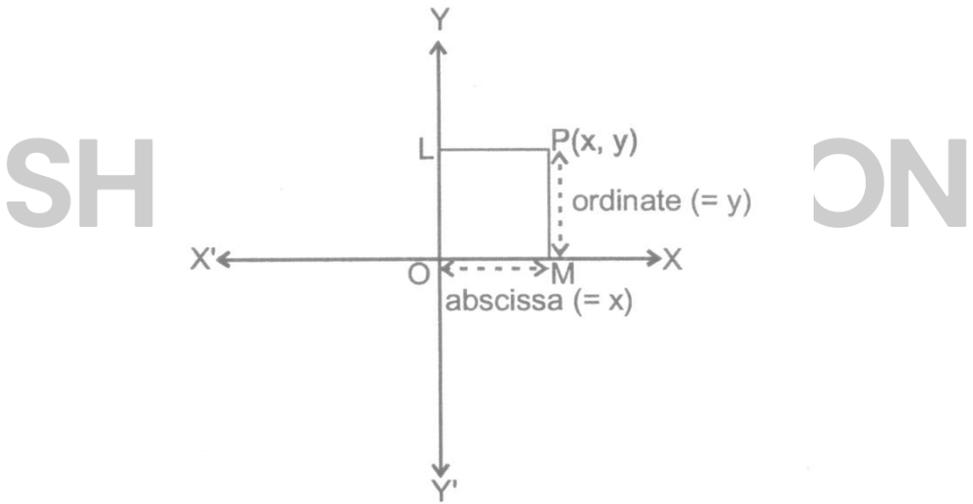
The two liens **X'OX** and **Y'OY** divide the plane into four parts called **quadrants. XOY, YOX' X'OY'** and **Y'OX** are, respectively, called the first, second third and fourth quadrants. The following table shows the signs of the coordinates of pins situated in different quadrants :

| Quadrant	X-coodrinate	Y-coordinate	Point
First quadrant	+	+	(+, +)
Second quadrant	-	+	(-, +)
Third quadrant	-	-	(-, -)
Fourth quadrant	+	-	(+, -)

Remaks

(i) Abscissa is the perpendicular distance of a point from **y-axis** (i.e., positive to the right of **y-axis** and negative to the left of **y - axis)**

(ii) **Ordinate** is positive above **x - axis** and negative below **x-axis.**

(iii) **Abscissa** of any point on **y-axis** is zero.

(iv) **Ordinate** of any point of **x-axis** is zero.

(v) Co-ordinates of the **origin** are (0,0)

Distance between two points:

Let two points be $P(x_1, y_1)$ and $Q(x_2, y_2)$

Take two mutually perpendicular lines as the coordinate axis with **O** as origin. Mark the points $P(x_1, y_1)$ and $Q(x_2, y_2)$. Draw lines **PA, QB** perpendicular to **X-axis** from the points **P** and **Q,** which meet the **X-axis** in points A and B, respectively.
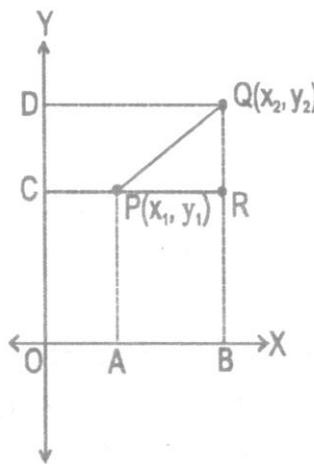
Draw lines PC and QD perpendicular to **Y-axis,** which meet the **Y-axis** in C and D, respectively. Produce CP to meet BQ in R. Now

OA = abscissa of P = x_1

Similarly, OB = x_2, OC = y_1 and OD = y_2

Therefore, we have

PR = AB = OB - OA = $x_2 - x_1$

Similarly, QR = QB - RB = QB - PA = $y_2 - y_1$

Now, using Pythagoras Theorem, in right angled triangle PRQ, we have

$PQ^2 = Pr^2 + RQ^2$

or $\quad PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$

Since the distance or length of the line-segment PQ is always non-negative, on taking the positive square root, we get the distance as

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

This result is known as **distance formula.**

Corollary : The distance of a point $\mathbf{P(x_1, y_1)}$ from the origin **(0,0)** is given by

$$OP = \sqrt{x_1^2 + y_1^2}$$

Some useful points :

1.In questions relating to geometrical figures, take the given vertices in the given order and proceed as indicated.

(i)   For an **isosceles triangle -** We have to prove that at least two sides are equal.

(ii)   For an **equilateral triangle -** We have to prove that three sides are equal.

(iii)   For a **right -angled triangle -** We have to prove that the sum of the squares of two sides is equal to the square of the third side.

(iv)   for a **square -** We have to prove that the four sides are equal, two diagonals are equal.

(v)   For a **rhombus -** We have to prove that four sides are equal (and there is no need to establish that two diagonals are unequal as the square is also a rhombus).

(vi)   For a **rectangle -** We have to prove that the opposite sides are equal and two diagonals are equal.

(vii)   For a **Parallelogram -** We have to prove that the opposite sides are equal (and there is no need to establish that two diagonals are unequal sat the rectangle is also a parallelogram).

2. for three points to be **collinear -** We have to prove that the sum of the distances between two pairs of points is equal to the third pair of points.

Example: Find the distance between the points (8 , -2) and (3, -6).

Sol.   Let the points (8, -2) and (3, -6) be denoted by P and Q, respectively.

Then, by distance formula, we obtain the distance PQ as

$$PQ = \sqrt{(3-8)^2 + (-6+2)^2}$$

$$= \sqrt{(-5)^2 + (-4)^2} = \sqrt{41} \text{ unit}$$

Example: Prove that the points $(1,-1)$, $\left(-\dfrac{1}{2}, \dfrac{1}{2}\right)$ and $(1, 2)$ are the vertices of an isosceles triangle.

Sol. Let the point $(1, -1)$, $\left(-\dfrac{1}{2}, \dfrac{1}{2}\right)$ and $(1, 2)$ be denoted by P, Q and R, respectively. Now

$$PQ = \sqrt{\left(-\dfrac{1}{2} - \right)^2 + \left(\dfrac{1}{2} + 1\right)^2} = \sqrt{\dfrac{18}{4}} = \dfrac{3}{2}\sqrt{2}$$

$$QR = \sqrt{\left(1 + \dfrac{1}{2}\right)^2 + \left(2 - \dfrac{1}{2}\right)^2} = \sqrt{\dfrac{18}{4}} = \dfrac{3}{2}\sqrt{2}$$

$$PR = \sqrt{(1-1)^2 + (2+1)^2} = \sqrt{9} = 3$$

From the above, we see that PQ = QR

$\therefore$ The triangle is isosceles.

Example: Using distance formula, show that the points $(-3, 2)$, $(1, -2)$ and $(9, -10)$ are collinear.

Sol. Let the given points $(-3, 2)$, $(1, -2)$ and $(9, -10)$ be denoted by A, B and C, respectively. Points A, B and C will be collinear, if the sum of the lengths of two line-segments is equal to the third.

Now, $AB = \sqrt{(1+3)^2 + (-2-2)^2} = \sqrt{16+16} = 4\sqrt{2}$

$BC = \sqrt{(9-1)^2 + (-10+2)^2} = \sqrt{64+64} = 8\sqrt{2}$

$AC = \sqrt{(9+3)^2 + (-10-2)^2} = \sqrt{144+144} = 12\sqrt{2}$

Since, $AB + BC = 4\sqrt{2} + 8\sqrt{2} = 12\sqrt{2} = AC$, the, points A, B and C are collinear.

Example: Find a point on the X-axis which is equidistant from the points $(5, 4)$ and $(-2, 3)$.

Sol. Since the required point (say P) is on the X-axis, its ordinate will be zero. Let the abscissa of the point be x.

Therefore, coordinates of the point P are $(x, 0)$.

Let A and B denote the points $(5, 4)$ and $(-2, 3)$, respectively.

Since we are given that AP = BP, we have

$AP^2 = BP^2$

i.e., $(x - 5)^2 + (0 - 4)^2 = (x + 2)^2 + (0 - 3)^2$

or $x^2 + 25 - 10x + 16 = x^2 + 4 + 4x + 9$

or $-14x = -28$

or $x = 2$

Thus, the required point is $(2, 0)$.

Example: The vertices of a triangle are (-2, 0) , (2, 3) and (1, -3). Is the triangle equilateral, isosceles or scalene?

Sol. Let the points (-2, 0), (2, 3) and (1, -3) be denoted by A, B and C respectively. Then,

$$AB = \sqrt{(2+2)^2 + (3-0)^2} = 5$$

$$BC = \sqrt{(1-2)^2 + (-3-3)^2} = \sqrt{37}$$

and $AC = \sqrt{(1+2)^2 + (-0-0)^2} = 3\sqrt{2}$

Clearly, $AB \neq BC \neq AC$.

Therefore, ABC is a scalene triangle.

Example: The length of a line-segments is 10. If one end is at (2, -3) and the abscissa of the second end is 10, show that its ordinate is either 3 or -9.

Sol. Let (2, -3) be the point A. let the ordinate of the second end B be y. Then its coordinates will be (10, y).

$\therefore$ $AB = \sqrt{(10-2)^2 + (y+3)^2} = 10$ (Given)

or $64 + 9 + y^2 + 6y = 100$

or $y^2 + 6y + 73 - 100 = 0$

or $y^2 + 6y - 27 = 0$

or $(y + 9)(y - 3) = 0$

Therefore, $y = 9$ or $y = 3$.

Example: Show that the points (-2, 5), (3, -4) and (7, 10) are the vertices of a right triangle.

Sol. Let the three points be A(-2, 5), B(3, -4) and C(7, 10).

Then $AB^2 = (3 + 2)^2 + (-4 - 5)^2 = 106$

$BC^2 = (7 - 3)^2 + (10 + 4)^2 = 212$

$AC^2 = (7 + 2)^2 + (10 - 5)^2 = 106$

We see that

$BC^2 = AB^2 1 + AC^2$

$212 = 106 + 106$

$212 = 212$

$\therefore$ $\angle A = 90^0$

Thus, ABC is a right triangle, right angled at A.

Example: If the distance of P (x, y) from A (5, 1) and B(-1, 5) are equal, prove that 3x = 2y.

Sol. P(x, y), A (5, 1) and B (-1, 5) are the given points.

AP = BP(Given)

$\therefore$ $\qquad$ $AP^2 = BP^2$

or $\qquad$ $AP^2 - BP^2 = 0$

or $\qquad$ $\{(x - 5)^2 + (y - 1)\}^2 - \{(x + 1)^2 + (y - 5)^2\} = 0$

or $\qquad$ $x^2 + 25 - 10x + y^2 + 1 - 2y - x^2 - 1 - 2x - y^2 - 25 + 10y = 0$

or $\qquad$ $-12x + 8y = 0$

or $\qquad$ $3xx = 2y.$

Section Formulae:

(a) $\qquad$ **Formula for Internal Division :**

The coordinates of the pint which divided the line segment joining the pints (x_1, y_1) and $x_2, y_2)$ internally in the ratio

m : n are given by $x = \dfrac{mx_2 + nx_1}{m + n}$, $y = \dfrac{my_2 + my_1}{m + n}$

Proof :

Let O be the origin and let OX and OY be the X-axis and Y-axis respectively. Let **A(x_1, y_1)** and **B(x_2, y_2)** bet the given points. Let **(x, y)** be the coordinates of the point **p** which divides **AB** internally in the ratio **m : n** Draw AL $\perp$ OX, BM $\perp$ OX, PN $\perp$ Ox. Also, draw **AH** and **PK** perpendicular from **A** and **P** on **PN** and **BM** respectively. Then

OL = x_1, ON = x, OM = x_2, AL = y_1, PN = y and BM = y_2.

$\therefore$ $\qquad$ AH = LN = ON - OL = x - x_1, PH = PH - HN

= PN - AL = y - y_1, PK = NM = OM - ON = x_2 - x

$\qquad$ and $\qquad$ BK = BM - MK = BM - PN = y_2 - y.

Clearly, $\triangle$AHP and $\triangle$PKB are similar.

$\therefore$ $\qquad$ $\dfrac{AP}{BP} = \dfrac{AH}{PK} = \dfrac{PH}{BK}$

$\Rightarrow$ $\qquad$ $\dfrac{m}{n} = \dfrac{x - x_1}{x_2 - x} = \dfrac{y - y_1}{y_2 - y}$

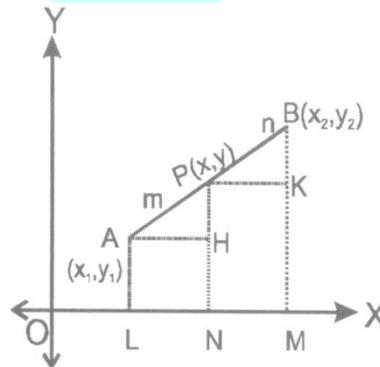Now, $\qquad$ $\dfrac{m}{n} = \dfrac{x - x_1}{x_2 - x}$

$\Rightarrow$ $\qquad$ $mx_2 - mx = nx - nx_1$ $\qquad\qquad$ $\Rightarrow$ $\qquad$ $mx + nx = mx_2 + nx_1$

$\Rightarrow$ $\qquad$ $x = \dfrac{mx_2 + nx_1}{m + n}$

and $\qquad$ $\dfrac{m}{n} = \dfrac{y - y_1}{y_2 - y}$

$\Rightarrow$ $\qquad$ $my_2 - my = ny - ny_1$ $\qquad\qquad$ $\Rightarrow$ $\qquad$ $my + ny = my_2 + ny_1$

$$\Rightarrow \quad y = \frac{my_2 + ny_1}{m + n}$$

Thus, the coordinates of P are $\left(\frac{mx_2 + nx_1}{m + n}, \frac{my_2 + ny_1}{m + n} \right)$

REMARKS

If **P** is the mid-point of **AB,** then it divides **AB** in the ratio **1 : 1,** so its coordinates are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$

(b)   Formula for External Division :

The coordinates of the points which divides the line segment joining the points (x_1, y_1) and (x_2, y_2) externally in the ratio **m : n** are given by

$$x = \frac{mx_2 - nx_1}{m - n}, y = \frac{my_2 - ny_1}{m - n}$$

Example: Find the coordinates of the point which divides the line segment joining the points (6, 3) and (-4, 5) in the ratio 3 : 2 (i) internally (ii) externally.

Sol.   Let P(x, y) be the required point.

(i) For internal division, we have

$$x = \frac{3x - 4 + 2 \times 6}{3 + 2}$$

and $y = \frac{3 \times 5 + 2 \times 3}{3 + 2}$

$\Rightarrow \quad x = 0$ and $y = \frac{21}{5}$
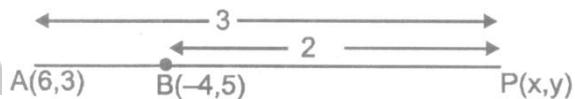
So the coordinates of P are (0, 21/5)

(ii) For external division, we have

$$x = \frac{3x - 4 - 2 \times 6}{3 - 2}$$

any $y = \frac{3 \times 5 - 2 \times 3}{3 - 2}$

$\Rightarrow \quad x = - 24$ and $y = 9$

So the coordinates of P are (-24, 9).

Example: In which ratio does the point (-1, -1) divides the line segment joining the pints (4, 4) and (7, 7) ?

Sol.   Suppose the point C(-1, -1) divides the line joining the points A(4, 4) and B(7, 7) in the ratio k : 1 Then, the coordinates

of C are $\left(\frac{7k + 4}{k + 1}, \frac{7k + 4}{k + 1} \right)$

But, we are given that the coordinates of the points C are (-1, -1).

$$\therefore \quad \frac{7k+4}{k+1} = -1 \Rightarrow k = -\frac{5}{8}$$

Thus, C divides AB externally in the ratio 5 : 8.

Example: In what ratio does the X-axis divide the line segment joining the points (2, -3) and (5, 6) ?

Sol. Let the required ratio be k : 1. Then the coordinates of the point of division are $\left(\frac{5\lambda+2}{k+1}, \frac{6\lambda-3}{k+1} \right)$. But, it is a point on X-axis on which y-coordinate of every point is zero.
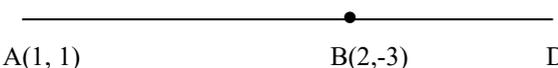
$$\therefore \quad \frac{6\lambda-3}{k+1} = 0$$

$$\Rightarrow \quad k = \frac{1}{2}$$

Thus, the required ratio is $\frac{1}{2}$: 1 or 1 : 2.

Example: A (1, 1) and B(2, -3) are two points and D is a point on AB produced such that AD = 3 AB. Find the coordinates of D.

Sol. We have, AD = 3AB. Therefore, BD = 2AB. Thus D divides AB externally in the ratio AD : BD = 3 : 2 Hence, the coordinates of D are

$$\therefore \quad \left(\frac{3 \times 2 - 2 \times 1}{3-2}, \frac{3x - 3 - 2 \times 1}{3-2} \right)$$

$$= \quad (4, -11).$$

A(1, 1)   B(2,-3)   D

Example: Determine the ratio in which the line 3x + y - 9 = 0 divides the segment joining the pints (1, 3) and (2, 7).

Sol. Suppose the line 3x + y - 9 = 0 divides the line segment joining A(1, 3) and B(2, 7) in the ratio k : 1 at point C. The, the coordinates of C are $\left(\frac{2k+1}{k+1}, \frac{7k+3}{k+1} \right)$ But, C lies on 3x + y - 9 = 0, therefore

$$3\left(\frac{2k+1}{k+1} \right) + \frac{7k+3}{k+1} - 9 = 0$$

$$\Rightarrow \quad 6k + 3 + 7k + 3 - 9k - 9 = 0$$

$$\Rightarrow \quad k = \frac{3}{4}$$

So, the required ratio is 3 : 4 internally.

Centroid of A Triangle:

Prove that the coordinates of the triangle whose vertices are (x_1, y_1), (x_2, y_2) and (y_3, y_3) are $\left(\dfrac{x_1 + x_2 + x_3}{3}, \dfrac{y_1 + y_2 + y_3}{3}\right)$. Also, deduce that the medians of a triangle are concurrent.

Proof :

Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of $\triangle ABC$ whose medians are AD, BE and CF respectively. So. D,E and F are respectively the mid-points of BC, CA and AB.

Coordinates of **D** are $\left(\dfrac{x_2 + x_3}{2}, \dfrac{y_2 + y_3}{2}\right)$. Coordinates of a point dividing AD in the ratio **2 : 1** are
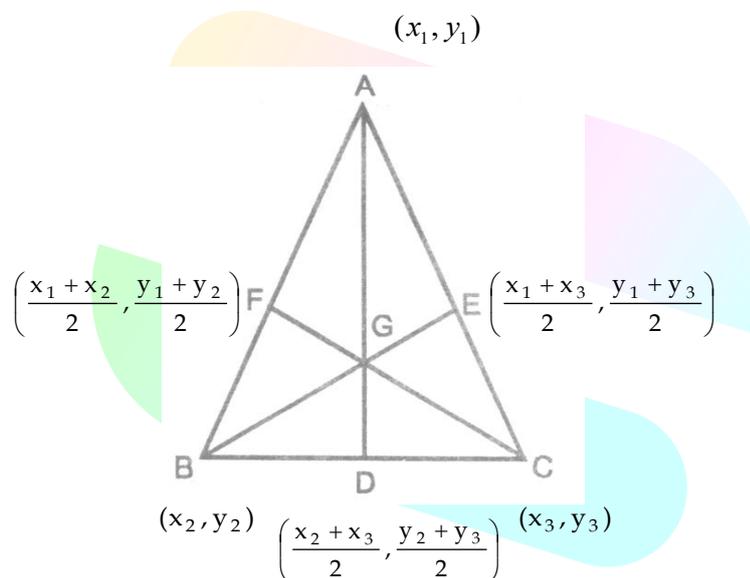
$$\left(\frac{1.x_1 + 2\left(\dfrac{x_2 + x_3}{2}\right)}{1+2}, \frac{1.y_1 + \left(\dfrac{y_2 + y_3}{2}\right)}{1+2}\right) = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

The coordinates of E are $\left(\dfrac{x_1 + x_3}{2}, \dfrac{y_1 + y_3}{2}\right)$. The coordinates of a point dividing BE in the ratio 2 : 1 are

$$\left(\frac{1.x_2 + \dfrac{2(x_1 + x_3)}{2}}{1+2}, \frac{1.y_2 + \dfrac{2(y_1 + y_3)}{2}}{1+2}\right) = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

Similarly the coordinates of a point dividing CF in the ratio 2 : 1 are $\left(\dfrac{x_1 + x_2 + x_3}{3}, \dfrac{y_1 + y_2 + y_3}{3}\right)$

Thus, the point having coordinates $\left(\dfrac{x_1 + x_2 + x_3}{3}, \dfrac{y_1 + y_2 + y_3}{3}\right)$ is common to AD, BE and CF and divides them in
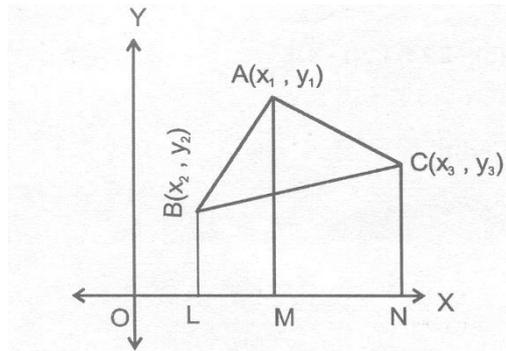
the ratio 1 : 2.

Hence, medians of a triangle are concurrent and the coordinates of the centroid are

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3} \right).$$

Area of A Triangle:

Let **ABC** be any triangle whose vertices are $\mathbf{A(x_1, y_1)}$ $\mathbf{B(x_2, y_3)}$. Draw BL, AM and CN perpendicular from B,A and C respectively, to the X-axis. ABLM, AMNC and BLNC are all trapeziums.

Area of ΔABC = Area of trapezium ABLM + Area of trapezium AMNC - Area of trapezium BLNC We know that, Area of trapezium $= \frac{1}{2}$ (Sum of parallel sides) (distance b/w them)

Therefore

Area of ΔABC $= \frac{1}{2}$ (BL + AM) (LM) + $\frac{1}{2}$ (AM + CN) MN- $\frac{1}{2}$ (BL + CN) (LN)

Area of ΔABC $= \frac{1}{2}$ $(y_2 + y_1)$ $x_1 - x_2)$ + $\frac{1}{2}$ $(y_1 + y_3)$ $(x_3 - x_1)$ - $\frac{1}{2}$ $(y_2 + y_3)$ $(x_3 - x_2)$

Area of ΔABC $= \frac{1}{2} \left| [x_1(y_2 - y_3) + x_2(y_3 - y\ ) + x_3(y_1 - y_2)] \right|$

(a)   Condition for collinearity :

Three points $\mathbf{A\ (x_1, y_1)}$ $\mathbf{B(x_2, y_2)}$ and $\mathbf{C(x_3, y_3)}$ are collinear if **Area of ΔABC = 0.**

Area of Quadrilateral:

Let the vertices of Quadrilateral ABCD are $\mathbf{A(x_1, y_1)}$, $\mathbf{B(x_2, y_2}$, $\mathbf{C(x_3, y_3)}$ and $\mathbf{D(x_4, y_4)}$

So, Area of quadrilateral ABCD = Area of Δ ABC + Area of Δ ACD
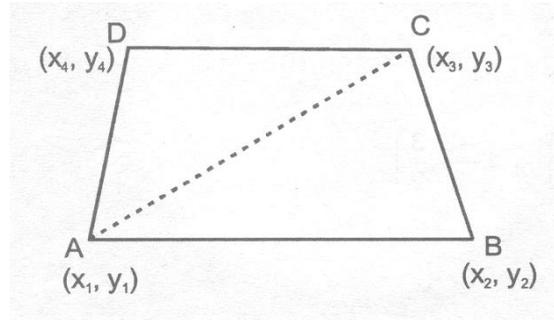
Example: The vertices of Δ ABC are (-2, 1), (5, 4) and (2, -3) respectively. Find the area of triangle.

Sol. A(-2, 1), B(-2, 1) and C(2, -3) be the vertices of triangle.

So, $x_1 = -2$, $y_1 = 1$; $x_2 = 5$, $y_2 = 4$; $x_3 = 2$ $y_3 = -3$

Area of Δ ABC $= \dfrac{1}{2}\left|[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]\right|$

$= \dfrac{1}{2}\left|[(-2)(4 + 3) + (5)(-3 - 1) + 2(1 - 4)]\right|$

$= \dfrac{1}{2}\left|[-14 + (-20) + (-6)]\right|$

$= \dfrac{1}{2}\left|-40\right|$

$= 20$ Sq. unit.

Example: The area of a triangle is 5. Two of its vertices area (2, 1) and (3, -2). The third vertex lies on y = x + 3. Find the third vertex.

Sol. Let the third vertex be (x_3, y_3) area of triangle

$= \dfrac{1}{2}\left|[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]\right|$

As $x_1 = 2$ $y_1 = 1$; $x_2 = 3$, $y_2 = -2$; Area of $\Delta = 5$ sq. unit

$\Rightarrow$ $5 = \dfrac{1}{2}\left|2(-2 - y_3) + 3(y_3 - 1) + x_3(1 + 2)\right|$

$\Rightarrow$ $10 = |3x_3 + y_3 - 7|$

$\Rightarrow$ $3x_3 + y_3 - 7 = \pm 10$

Taking positive sign

$3x_3 + y_3 - 7 = 10 \Rightarrow$ $3x \therefore + y_3 = 17$ .....(i)

Taking negative sing

$\Rightarrow \quad 3x_3 + y_3 - 7 = -10$

$\Rightarrow \quad 3x\therefore + y_3 = -3$ .....(ii)

Given that (x_3, y_3) lies on $y = x + 3$

So, $\quad -x\therefore + y_3 = 3$ ....(iii)

Solving eq. (i) & (iii)

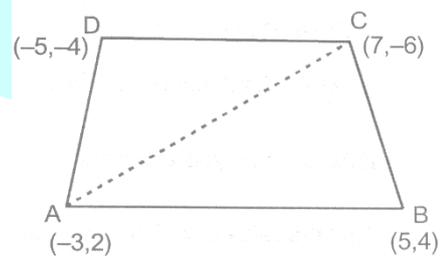$$x_3 = \frac{7}{2}, \qquad\qquad y_3 = \frac{13}{2}$$

Solving eq. (ii) & (iii)

$$x_3 = \frac{-3}{2}, \qquad\qquad y_3 = \frac{3}{2}$$

So the third vertex are $\left(\frac{7}{2}, \frac{13}{2}\right)$ or $\left(\frac{-3}{2}, \frac{3}{2}\right)$

Example: Find the area of quadrilateral whose vertices, taken in order, are (-3, 2), B(5, 4), (7, -6) and D(-5, -4).

Sol. Area of quadrilateral = Area of ΔABC + Area of Δ ACD

So, Area of ΔABC $= \frac{1}{2}\left|(-3)(4+6) + 5(-6-2) + 7(2-4)\right|$

$= \frac{1}{2}\left|-30 - 40 - 14\right|$

$= \frac{1}{2}\left|-84\right| = 42$ Sq. units

Area of Δ ACD $= \frac{1}{2}\left|-3(-6+4) + 7(-4-2) + (-5)(2+6)\right|$

$= \frac{1}{2}\left|+6 - 42 - 40\right| = \frac{1}{2}\left|-76\right| = 38$ Sq. units

So, Area of quadrilateral ABCD = 42 + 38 = 80 Sq. units.