SHIKSHA NATION

Quadratic Equations

Quadratic Equation:

If P(x) is quadratic expression in variable x, then P(x) = 0 is known as a quadratic equation.

(a) General form of a Quadratic Equation :

The general form of quadratic equation is $\mathbf{ax^2 + bx + c = 0}$, where a,b,c are real numbers and $a \neq 0$ Since $a \neq 0$, quadratic equations, in general are of the following types :-

(i) $b = 0,\ c \neq 0$ i.e., of he type $ax^2 + c=0$.

(ii) $b \neq 0$, $c = 0$, i.e. of the type $ax^2 + bx = 0$.

(iii) $b = 0$, $c = 0$, i.e. of the type $ax^2 = 0$.

(iv) $b \neq 0$, $c \neq 0$, i.e., of the type $ax^2 + bx + c = 0$.

Roots of A Quadratic Equation:

The value of x which satisfies the given quadratic equation is known as its root. The roots of the given equation are known as its solution.

General form of a quadratic equation is :

$$ax^2 + bx + c = 0$$

or $4a^2x^2 + 4abx + 4ac = - 4ac$ [Multiplying by 4a]

or $4a^2x^2 + 4abx = - 4ac$ [By adding b^2 both sides]

or $4a^2x^2 + 4abc + b^2 = b^2 - 4ac$

or $(2ax + b)^2 = b^2 - 4ac$

Taking square root of both the sides

$$2ax + b = \pm\sqrt{b^2 - 4ac}$$

or $$x = \frac{-b \pm \sqrt{b2 - 4ac}}{2a}$$

Hence, roots of the quadratic equation $ax^2 + bx + c = 0$ are $\frac{-b+\sqrt{b^2-4ac}}{2a}$ and $\frac{-b-\sqrt{b^2-4ac}}{2a}$

REMARK:

A quadratic equation is satisfied by exactly two values of 'a' which may be real or imaginary. The equation, $ax^2 + bx + c = 0$ is :

- ★ A quadratic equation if $a \neq 0$ Two roots
- ★ A linear equation if $a = 0,\ b \neq 0$ One root
- ★ A contradiction if $a = b = 0,\ c \neq 0$ No root
- ★ An identify if $a = b = c = 0$ Infinite roots
- ★ A quadratic equation cannot have more than two roots.
- ★ If follows from the above statement that if a quadratic equation is satisfied by more than two values of x, then it is satisfied by every value of x and so it is an identity.

Nature of Roots:

Consider the quadratic equation, $ax^2 + bx + c = 0$ having $\alpha\ \beta$ as its roots and $b^2 - 4ac$ is called discriminate of roots of quadratic equation. It is denoted by D or Δ.

Roots of the given quadratic equation may be

(i) Real and unequal (ii) Real and equal (iii) Imaginary and unequal.

Let the roots of the quadratic equation $ax^2 + bx + c = 0$ (where $a \neq 0, b, c \in R$) be α and β then

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad .....(i)$$

and $$\beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \quad ....(ii)$$

The nature of roots depends upon the value of expression **'b^2 - 4ac'** with in the square root sign. This is known as discriminate of the given quadratic equation.

Consider the Following Cases :

Case-1 When $b^2 - 4ac > 0$, (D > 0)

In this case roots of the given equation are real and distinct and are as follows

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

(i) When $a(\neq 0), b, c \in Q$ and b^2 - 4ac is a perfect square

In this case both the roots are rational and distinct.

(ii) When $a(\neq 0), b, c \in Q$ and b^2 - 4ac is not a perfect square

In this case both the roots are irrational and distinct. **[See remarks also]**

Case-2 When $b^2 - 4ac = 0$, (D = 0)

In this case both the roots are real and equal to $-\frac{b}{2a}$.

Case-3 When b^2 - 4ac < 0, (D < 0)

In this case $b^2 - 4ac < 0$, then $4ac - b^2 > 0$

$$\therefore \quad \alpha = \frac{-b+\sqrt{-(4ac-b^2)}}{2a} \text{ and } \beta = \frac{-b-\sqrt{(4ac-b^2)}}{2a}$$

$$\text{or} \quad \alpha = \frac{-b+i\sqrt{4ac-b^2}}{2a} \text{ and } \beta = \frac{-b-i\sqrt{4ac-b^2}}{2a} \qquad \left[\because \sqrt{-1} = i\right]$$

i.e. in this case both the root are imaginary and distinct.

Remarks:

★ If **a,b,c** $\in$ **Q** and **b^2 - 4ac** is positive **(D > 0)** but not a perfect square, then the roots are irrational and they always occur in conjugate pairs like $2+\sqrt{3}$ and $2-\sqrt{3}$. However, if **a,b,c** are irrational number and **b^2 - 4ac** is positive but not a perfect square, then the roots may not occur in conjugate pairs.

★ If **b^2 - 4ac** is negative **(D > 0)**, then the roots are complex conjugate of each other. In fact, complex roots of an equation with real coefficients always occur in conjugate pairs like 2 + 3i and 2 - 3i. However, this may not be true in case of equations with complex coefficients. For example, $x^2 - 2ix - 1 = 0$ has both roots equal to i.

★ If **a** and **c** are of the same sign and b has a sign opposite to that of a as well as c, then both the roots are positive, the sum as well as the product of roots is positive $(D \geq 0)$.

★ If **a,b,** are of the same sign then both the roots are negative, the sum of the roots is negative but the product of roots is positive $(D \geq 0)$.

Methods of Solving Quadratic Equation:

(a) By Factorisation :

Algorithm:

Step (i) Factorise the constant term of the given quadratic equation.

Step (ii) Express the coefficient of middle term as the sum or difference of the factors obtained in step 1. Clearly, the product of these two factors will be equal to the product of the coefficient of x^2 and constant term.

Step (iii) Split the middle term in two parts obtained in step 2.

Step (iv) Factorise the quadratic equation obtained in step 3.

Illustrations:

Example: Solve the following quadratic equation by factorisation method: $x^2 - 2ax + a^2 - b^2 = 0$.

Sol. Here, Factors of constant term $(a^2 - b^2)$ are (a - b) and (a + b).

Also, Coefficient of the middle term = - 2a = - [(a - b) + (a + b)]

$\therefore \quad x^2 - 2ax + a^2 - b^2 = 0$

$\Rightarrow \quad x^2 - \{(a - b) + (a + b)\}x + (a - b)(a + b) = 0$

$\Rightarrow \quad x^2 - (a - b)x - (a + b)x + (a - b)(a + b) =$

$\Rightarrow \quad x\{x - (a - b)\} - (a + b)\{x - (a - b)\} = 0$

$\Rightarrow$ $\{x - (a - b)\}\ \{x - (a + b)\} = 0$

$\Rightarrow$ $x - (a - b) = 0$ or, $x - (a + b) = 0$

$\Rightarrow$ $x = a - b$ or $x = a + b$

Example: Solve $64x^2 - 625 = 0$

Sol. We have $64x^2 - 625 = 0$

or $(8x)^2 - (25)^2 = 0$

or $(8x + 25)\ (8x - 25) = 0$

i.e. $8x + 25 = 0$ o $8x - 25 = 0$.

This gives $x = \frac{25}{8}$ or $\frac{25}{8}$.

Thus, $x = -\frac{25}{8}, \frac{25}{8}$ are solutions of the given equations.

Example: Solve the quadratic equation $16x^2 - 24x = 0$.

Sol. The given equation may be written as $8x(2x - 3) = 0$

This gives $x = 0$ or $x = \frac{3}{2}$.

$x = 0, \frac{3}{2}$, are the required solutions.

Example: Solve :- $25x^2 - 30x + 9 = 0$

Sol. $25x^2 - 30x + 9 = 0$ is equivalent to $(5x)^2 - 2(5x) \times 3 + (3)^2 = 0$

or $(5x - 3)^2 =$

This gives $x = \frac{3}{5}, \frac{3}{5}$ or simply $x = \frac{3}{5}$ as the required solution.

Example: Find the solutions of the quadratic equation $x^2 + 6x + 5 = 0$.

Sol. The quadratic polynomial $x^2 + 6x + 5$ can be facorised as follows :-

$x^2 + 6x + 5 = x^2 + 5x + x + 5$

$= x(x + 5) + 1\ (x + 5)$

$= (x + 5)\ (x + 1)$

Therefore the given quadratic equation becomes $(x + 5)\ (x + 1) =$

This gives $x = -5$ or $= -1$

Therefore, $x = -1$ are the required solutions of the given equation.

Example: Solve : $\frac{2x}{x-3}+\frac{1}{2x+3}+\frac{3x+9}{(x-3)(2x+3)}=0.$

Sol. Obviously, the given equation is valid if $x - 3 \neq 0$ and $2x + 3 \neq 0$.

Multiplying throughout by (x - 3) (2x - 3), we get

$2x(2x + 3) + 1(x - 3) + 3x + 9 = 0$

or $4x^2 + 10 + 6 = 0$

or $2x^2 + 5x + 3 = 0$

or $(2x + 3)(x + 1) = 0$

But $2x + 3 \neq 0$, so we get $x + 1 = 0$.

This gives $x = -1$ as the only solution of the given equation.

(b) By the Method of Completion of Square :

Algorithm:

Step-(i) Obtain the quadratic equation. Let the quadratic equation be $ax^2 + bx + c = 0, a \neq 0$.

Step-(ii) Make the coefficient of x^2 unity, if it is not unity. i.e., obtained $x^2+\frac{b}{a}x+\frac{c}{a}=0.$

Step-(iii) Shift the constant term $\frac{c}{a}$ on R.H.S. to get $x^2+\frac{b}{a}x=-\frac{c}{a}$

Step-(iv) Add square of half of the coefficient of x i.e. $\left(\frac{b}{2a}\right)^2$ on both sides to obtain

$$x^2+2\left(\frac{b}{2a}\right)x+\left(\frac{b}{2a}\right)^2=\left(\frac{b}{2a}\right)^2-\frac{c}{a}$$

Step-(v) Write L.H.S. as the perfect square of a binomial expression and simplify R.H.S. to get

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}.$$

Step-(vi) Take square root of both sides to get $x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}$

Step (vii) Obtain the values of x by shifting the constant term $\frac{b}{2a}$ on RHS.

Example: Solve :- $x^2 + 3x + 1 = 0$

Sol. We have

$x^2 + 3x + 1 = 0$

Add and subtract ($\frac{1}{2}$ coefficient of x)^2 in L.H.S. and get

$$x^2 + 3x + 1 + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 = 0$$

$$\Rightarrow \quad x^2 + 2\left(\frac{3}{2}\right)x + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 + 1 = 0$$

$$\Rightarrow \quad \left(x + \frac{3}{2}\right)^2 - \frac{5}{4} = 0$$

$$\Rightarrow \quad \left(x + \frac{3}{2}\right)^2 = \left(\frac{\sqrt{5}}{2}\right)^2$$

$$\Rightarrow \quad x + \frac{3}{2} = \pm\frac{\sqrt{5}}{2}$$

This gives $x = \frac{-(3+\sqrt{5})}{2}$ or $x = \frac{-3+\sqrt{5}}{2}$

Therefore $x = -\frac{3+\sqrt{5}}{2}, \frac{-3+\sqrt{5}}{2}$ are the solutions of the given equation.

Example: By using the method of completing the square, show that the equation $4a^2 + 3x + 5 = 0$ has no real roots.

Sol. We have, $4x^2 + 3x + 5 = 0$

$$\Rightarrow \quad x^2 + \frac{3}{4}x + \frac{5}{4} = 0$$

$$\Rightarrow \quad x^2 + 2\left(\frac{3}{8}x\right) = -\frac{5}{4}$$

$$\Rightarrow \quad x^2 + 2\left(\frac{3}{8}\right)x + \left(\frac{3}{8}\right)^2 = \left(\frac{3}{8}\right)^2 - \frac{5}{4}$$

$$\Rightarrow \quad \left(x + \frac{3}{8}\right)^2 = -\frac{71}{64}$$

Clearly, RHS is negative

But, $\left(x+\frac{3}{8}\right)^2$ cannot be negative for any real value of x.

Hence, the given equation has no real roots.

(c) By Using Quadratic Formula :

Solve the quadratic equation in general form viz. $ax^2 + bx + c = 0$.

We have, $ax^2 + bx + c = 0$

Step (i) By comparison with general quadratic equation, find the value of a,b and c.

Step (ii) Find the discriminate of the quadratic equation.

$$D = b^2 - 4ac$$

Step (iii) Now find the roots of the equation by given equation

$$x = \frac{-b+\sqrt{D}}{2a}, \frac{-b-\sqrt{D}}{2a}$$

Remark:

★ If $\mathbf{b^2 - 4ac < 0}$ i.e. negative, then $\sqrt{b^2 - 4ac}$ is not real and therefore, the equation does not have any real roots.

Example: Solve the quadratic equation $x^2 - 7x - 5 = 0$.

Sol. Comparing the given equation with $ax^2 + bx + c = 0$, we find that a = 1, b = - 7 and c = -5.

Therefore, $D = (-7)^2 - 4 \times 1 \times (-5) = 49 + 20 = 69 > 0$

Since D is positive, the equation has two roots given by $\frac{7+\sqrt{69}}{2}, \frac{7-\sqrt{69}}{2}$

$\Rightarrow$ $x = \frac{7+\sqrt{69}}{2}, \frac{7-\sqrt{69}}{2}$ are the required solutions.

Example: For what value of k, $(4 - k)x^2 + (2k + 4)x + (8k + 1)$ is a perfect square.

Sol. The given equation is a perfect square, if its discriminate is zero i.e. $(2k + 4)^2 - 4(4 - k)(8k + 1) = 0$

$\Rightarrow$ $4(k + 2)^2 - 4(4 - k)(8k + 1) = 0 \Rightarrow 4[4(k + 2)^2 - (4 - k)(8k + 1)] = 0$

$\Rightarrow$ $[(k^2 + 4k + 4) - (-8k^2 + 31k + 4)] = 0 \Rightarrow 9k^2 - 27k = 0$

$\Rightarrow$ $9k(k - 3) = 0 \Rightarrow k = 0$ or $k = 3$

Hence, the given equation is a perfect square, if k = 0 or k = 3.

Example: If the roots of the equation $a(b - c)x^2 + b(c - a)x + c(a - b) = 0$ are equal, show that $\frac{2}{b} = \frac{1}{a} + \frac{1}{c}$.

Sol. Since the roots of the given equations are equal, so discriminant will be equal to zero.

$\Rightarrow$ $b^2(c - a)^2 - 4a(b - c) . c(a - b) = 0$

$\Rightarrow$ $b^2(c^2 + a^2 - 2ac) - 4ac(ba - ca - b^2 + bc) = 0,$

$\Rightarrow$ $a^2b^2 + b^2c^2 + 4a^2c^2 + 2b^2ac - 4ac^2bc - 4abc^2 = 0 \Rightarrow (ab + bc - 2ac)^2 = 0$

$\Rightarrow$ $ab + bc - 2ac = 0$

$\Rightarrow$ $ab + bc = 2ac$

$\Rightarrow$ $\frac{1}{c} + \frac{1}{a} = \frac{2}{b}$

$\Rightarrow$ $\frac{2}{b} = \frac{1}{a} + \frac{1}{c}$. **Hence Proved.**

Example: If the roots of the equation $(b - c)x^2 + (c - a) x + (a - b) = 0$ are equal, then prove that $2b = a + c$.

Sol. If the roots of the given equation are equal, then discriminant is zero i.e.

$(c - a)^2 - 4(b - c) (a - b) = 0 \Rightarrow c^2 + a^2 - 2ac + 4b^2 - 4ab + 4ac - 4bc = 0$

$\Rightarrow$ $c^2 + a^2 + 4b^2 + 2ac - 4ab - 4bc = 0$

$\Rightarrow$ $(c + a - 2b)^2 = 0$

$\Rightarrow$ $c + a = 2b$ **Hence Proved.**

Example: If the roots of the equation $x^2 - 8x + a^2 - 6a = 0$ are real and distinct, then find all possible values of a.

Sol. Since the roots of the given equation are real and distinct, we must have $D > 0$

$\Rightarrow$ $64 - 4 (a^2 - 6a) > 0$

$\Rightarrow$ $4[16 - a^2 + 6a] > 0$

$\Rightarrow$ $-4(a^2 - 6a - 16) > 0$

$\Rightarrow$ $a^2 - 6a - 16 < 0$

$\Rightarrow$ $(a - 8) (a + 2) < 0$

$\Rightarrow$ $- 2 < a < 8$

Hence, the roots of the given equation are real if **'a'** lies between -2 and 8.

Applications of Quadratic Equations:

Algorithm:

The method of problem solving consist of the following three steps :

Step (i) Translating the word problem into symbolic language (mathematical statement) which means identifying relationship existing in the problem and then forming the quadratic equation.

Step (ii) Solving the quadratic equation thus formed.

Step (iii) Interpreting the solution of the equation, which means translating the result of mathematical statement into verbal language.

REMARKS:

★ Two consecutive odd natural numbers be $2x - 1, 2x + 1$ where $x \in N$
★ Two consecutive even natural numbers be $2x, 2x + 2$ where $x \in N$
★ Two consecutive even positive integers be $2x, 2x + 2$ where $x \in Z^+$
★ Consecutive multiples of 5 be $5x, 5x + 5, 5x + 10$

Example: The sum of the squares of two consecutive positive integers is 545. Find the integers.

Sol. Let x be one of the positive integers. Then the other integer is $x + 1$, $x \in Z^+$

Since the sum of the squares of the integers is 545, we get

$x^2 + (x + 1)^2 = 545$

or $2x^2 + 2x - 544 = 0$

or $x^2 + x - 272 = 0$

$x^2 + 17x - 16x - 272 = 0$

or $x(x + 17) - 16(x + 17) = 0$

or $(x - 16)(x + 17) = 0$

Here, $x = 16$ or $x = -17$ But, x is a positive integer. Therefore, reject $x = -17$ and take **x = 16.** Hence, two consecutive positive integers are 16 and $(16 + 1)$, i.e., 16 and 17.

Example: The length of a hall is 5 m more than its breath. If the area of the floor of the hall is 84 m^2, what are the length and the breadth of the hall ?

Sol. Let the breadth of the hall be x metres.

Then the length of the ball is $(x + 5)$ metres.

The area of the floor = $x(x + 5)$ m^2

Therefore, $x(x + 5) = 84$

or $x^2 + 5x - 84 = 0$

or $(x + 12)(x - 7) = 0$

This given $x = 7$ or $x = -12$.

Since, the breadth of the hall cannot be negative, we reject $x = -12$ and take x = - only.

Thus, breadth of the hall = 7 metres, and length of the hall = $(7 + 5)$, i.e., 12 metres.

Example: Out of group of swans $\frac{7}{2}$ times the square root of the total number are playing on the shore of a tank. The two remaining ones are playing, in deep water. What is the total number of swans ?

Sol. Let us denote the number of swans by x.

Then, the number of swans playing on the shore of the tank $= \frac{7}{2}\sqrt{x}$.

There are two remaining swans.

Therefore, $x = \frac{7}{2}\sqrt{x} + 2$

or $x - 2 = \frac{7}{2}\sqrt{x}$

or $(x-2)^2 = \left(\frac{7}{2}\right)^2 x$

or $4(x^2 - 4x + 4) = 49x$

or $4x^2 - 65x + 16 = 0$

or $4x^2 - 64x - x + 16 = 0$

or $4x(x - 16) - 1(x - 16) = 0$

or $(x - 16)(4x - 1) = 0$

This gives $x = 16$ or $x = \frac{1}{4}$

We reject $x = \frac{1}{4}$ and take $x = 16$.

Hence, the total number of swans is 16.

Example: The hypotenuse of a right triangle is 25 cm. The difference between the lengths of the other two sides of the triangle is 5 cm. Find the lengths of these sides.

Sol. Let the length of the shorter side b x cm. Then, the length of the longer side = (x + 5) cm.

Since the triangle is right-angled, the sum of the squares of the sides must be equal to the square of the hypotenuse (Pythagoras Theorem).

$x^2 + (x + 5)^2 = 235^2$

or $x^2 + x^2 + 10x + 25 = 625$

or $2x^2 + 10x - 600 = 0$

or $x^2 + 5x - 300 = 0$

or $(x + 20)(x - 15) = 0$

This gives $x = 15$ or $x = -20$

We reject x = - 20 and take x = 15.

Thus, length of shorter side = 15 cm.

Length of longer side = (15 + 5) cm, i.e., 20 cm.

Example: Swati can row her boat at a speed of 5 km/h in still water. If it takes her 1 hour more to row the boat 5.25 km upstream than to return downstream, find the speed of the stream.

Sol. Let the speed of the stream be x km/h

∴ Speed of the boat in upstream = (5 - x)km/h

Speed of the boat in downstream = (5 + x)km/h

Time, say t_1 (in hours), for going 5.25 km upstream = $\frac{5.25}{5-x}$

Time, say t_2 (in hours), for returning 5.25 km downstream = $\frac{5.25}{5+x}$

Obviously $t_1 > t_2$

Therefore, according to the given condition of the problem,

$t_1 = t_2 + 1$

i.e., $\frac{5.25}{5-x} = \frac{5.25}{5+x} + 1$

or $\frac{21}{4}\left(\frac{1}{5-x} - \frac{1}{5+x}\right) = 1$

or $21\left(\frac{5+x-5+x}{25-x^2}\right) = 4$

or $42x = 100 - 4x^2$

or $4x^2 + 42x - 100 = 0$

or $2x^2 + 21x - 50 = 0$

or $(2x + 25)(x - 2) = 0$

This gives x = 2, since we reject $x = \frac{-25}{2}$.

Thus, the speed of the stream is 2 km/h.

Example: The sum of the square of two positive integers is 208. If the square of the larger number is 18 times the smaller number, find the numbers. **[CBSE - 2007]**

Sol Let x be the smaller number.

Then, square of the larger number will be 18x.

Therefore, $x^2 + 18x = 208$

or $x^2 + 18x - 208 = 0$

or $(x - 8)(x + 26) = 0$

This gives x = 8 or x = - 26

Since the numbers are positive integers, we reject x = - 26 and take x = 8.

Therefore, square of larger number = 18 × 8 = 144.

So, larger number = $\sqrt{144} = 12$

Hence, the larger number is 12 and the smaller is 8.

Example: The sum 'S' of first n natural number is given by the relation $S = \frac{n(n+1)}{2}$. Find n, if the sum is 276.

Sol. We have

$$S = \frac{n(n+1)}{2} = 276$$

or $n^2 + n - 552 = 0$

This gives $n = \frac{-1+\sqrt{1+2208}}{2}, \frac{-1-\sqrt{1+2208}}{2}$

or $n = \frac{-1+\sqrt{2209}}{2}, \frac{-1-\sqrt{2209}}{2}$

or $n = \frac{-1+47}{2}, \frac{-1-47}{2}$

or n = 23, -24

We reject n = - 24, since -24 is not a natural number.

Therefore, n = 23.