



Polynomials

Polynomials:

An algebraic expression $f(x)$ of the form $f(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$, where $a_0, a_1, a_2, \dots, a_n$ are real numbers and all the index of x are non-negative integers is called polynomials in x and the highest Index n in called the degree of the polynomial, if $a_n \neq 0$.

(a) Zero Degree Polynomial :

Any non-zero number is regarded as a polynomial of degree zero or zero degree polynomial. For example, $f(x) = a$, where $a \neq 0$ is a zero degree polynomial, since we can write $f(x) = a$ as $f(x) = ax^0$.

(b) Constant Polynomial :

A polynomial of degree zero is called a constant polynomial. For example, $f(x) = 7$.

(c) Linear Polynomial :

A polynomial of **degree 1** is called a linear polynomial.

For example : $p(x) = 4x - 3$ and $f(t) = \sqrt{3}t + 5$ are linear polynomials.

(d) Quadratic Polynomial :

A polynomial of **degree 2** is called quadratic polynomial.

For example : $f(x) = 2x^2 + 5x - \frac{3}{5}$ and $g(y) = 3y^2 - 5$ are quadratic polynomials with real coefficients.

Important Formulae:

$$(x + a)^2 = x^2 + 2ax + a^2$$

$$(x - a)^2 = x^2 - 2ax + a^2$$

$$x^2 - a^2 = (x + a)(x - a)$$

$$x^3 + a^3 = (x + a)(x^2 - ax + a^2) = (x + a)^3 - 3xa(x + a)$$

$$x^3 - a^3 = (x - a)(x^2 + ax + a^2) = (x - a)^3 + 3xa(x - a)$$

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$$



$$(a + b)^3 = a^3 + b^3 + 3ab(a + b)$$

$$(a - b)^3 = a^3 - b^3 - 3ab(a - b)$$

$$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$$

Special Case : If $a + b + c = 0$ then $a^3 + b^3 + c^3 = 3abc$.

Graph of Polynomials:

In algebraic or in set theoretic language the graph of a polynomial $f(x)$ is the collection (or set) of all points (x, y) , where $y = f(x)$. In geometrical or in graphical language the graph of a polynomial $f(x)$ is a smooth free hand curve passing through points $(x_1, y_1), (x_2, y_2), (x_3, y_3), \dots$ etc. where y_1, y_2, y_3, \dots are the values of the polynomial $f(x)$ at x_1, x_2, x_3, \dots respectively.

In order to draw the graph of a polynomial $f(x)$, follow the following algorithm.

Algorithm:

Step (i) Find the values y_1, y_2, \dots, y_n of polynomial $f(x)$ on different points $x_1, x_2, \dots, x_n, \dots$ and prepare a table that gives values of y or $f(x)$ for various values of x .

x:	x_1	x_2	x_n	x_{n+1}
y = f(x)	$y_1=f(x_1)$	$y_2=f(x_2)$	$y_n=f(x_n)$	$y_{n+1} = f(x_{n+1})$

Step (ii) Plot that points $(x_1, y_1), (x_2, y_2), (x_3, y_3), \dots, (x_n, y_n), \dots$ on rectangular co-ordinate system. In plotting these points use different scales on the X and Y axes.

Step (iii) Draw a free hand smooth curve passing through points plotted in **step 2** to get the graph of the polynomial $f(x)$.

(a) Graph of a Linear Polynomial :

Consider a linear polynomial $f(x) = ax + b$, $a \neq 0$ Graph of $y = ax + b$ is a straight line. That in why $f(x) = ax + b$ is called a linear polynomial. Since two points determine a straight line, so only two points need to plotted to draw the line $y = ax + b$. The line represented by $y = ax + b$ crosses the X-axis at exactly one point, namely $(-\frac{b}{a}, 0)$.

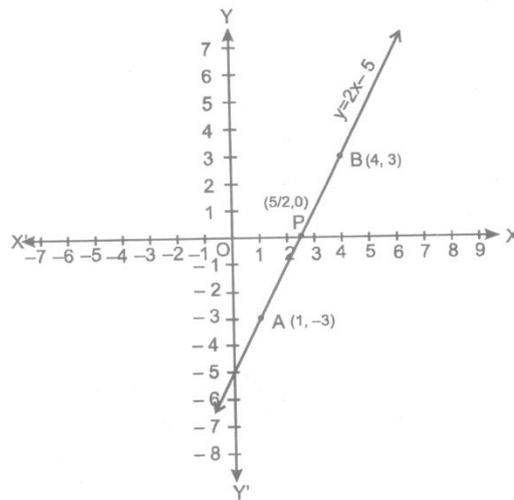
Examples: Draw the graph of the polynomial $f(x) = 2x - 5$. Also, find the coordinates of the point where it crosses X-axis.

Sol. Let $y = 2x - 5$.

The following table list the values of y corresponding to different values of x .

x	1	4
y	-3	3

The points A (1, - 3) and B (4, 3) are plotted on the graph paper on a suitable scale. A line is drawn passing through these points to obtain the graphs of the given polynomial.



(b) Graph of a Quadratic Polynomial :

Let a, b, c be real numbers and $a \neq 0$. Then $f(x) = ax^2 + bx + c$ is known as a quadratic polynomial in x . Graph of the quadratic polynomial i.e. the curve whose equation is $y = ax^2 + bx + c$, $a \neq 0$. Graph of a quadratic polynomial is always a parabola.

Let $y = ax^2 + bx + c$, where $a \neq 0$

$$\Rightarrow 4ay = 4a^2x^2 + 4abx + 4ac$$

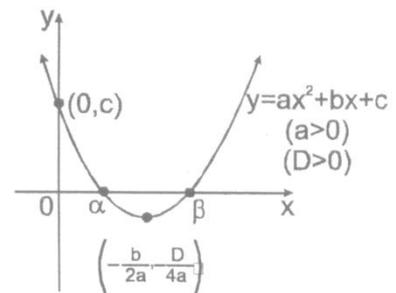
$$\Rightarrow 4ay = 4a^2x^2 + 4abx + b^2 - b^2 + 4ac$$

$$\Rightarrow 4ay = (2ax + b)^2 - (b^2 - 4ac)$$

$$\Rightarrow 4ay + (b^2 - 4ac) = (2ax + b)^2 \Rightarrow 4ay + (b^2 - 4ac) = 4a^2\left(x + \frac{b}{2a}\right)^2$$

$$\Rightarrow 4a\left\{y + \frac{b^2 - 4ac}{4a}\right\} = 4a^2\left(x + \frac{b}{2a}\right)^2$$

$$\Rightarrow \left(y + \frac{D}{4a}\right) = a\left(x + \frac{b}{2a}\right)^2 \dots(i)$$



where $D = b^2 - 4ac$ is the discriminant of the quadratic equation.

Remarks:

Shifting the origin at $\left(-\frac{b}{2a}, -\frac{D}{4a}\right)$, we have $X = x - \left(-\frac{b}{2a}\right)$ and $Y = y - \frac{(-D)}{4a}$

Substituting these values in (i), we obtain

$$Y = aX^2 \dots(ii)$$

which is the standard equation of parabola

Clearly, this is the equation of a parabola having its vertex at $\left(-\frac{b}{2a}, \frac{D}{4a}\right)$.

The parabola opens upwards or downwards according as $a > 0$ or $a < 0$.



Sign of Quadrative Expressions:

Let α be a real root of $ax^2 + bx + c = 0$. Then $a\alpha^2 + b\alpha + c = 0$. Point $(\alpha, 0)$ lies on $y = ax^2 + bx + c$. Thus, every real root of $ax^2 + bx + c = 0$ represents a point of intersecting of the parabola with the X-axis.

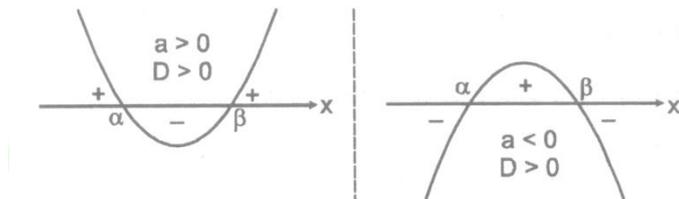
Conversely, if the parabola $y = ax^2 + bx + c$ intersects the X-axis at a point $(\alpha, 0)$ then $(\alpha, 0)$ satisfies the equation $y = ax^2 + bx + c$

$\Rightarrow a\alpha^2 + b\alpha + c = 0$ [α is a real root of $ax^2 + bx + c = 0$]

Thus, the intersection of the parabola $y = ax^2 + bx + c$ with X-axis gives all the real roots of $ax^2 + bx + c = 0$. Following conclusions may be drawn :-

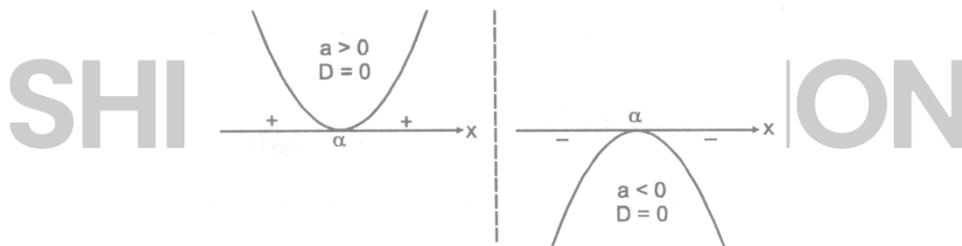
- (i) If $D > 0$, the parabola will intersect the x-axis in two distinct points and vice-versa.

The parabola meets x-axis at $\alpha = \frac{-b - \sqrt{D}}{2a}$ and $\beta = \frac{-b + \sqrt{D}}{2a}$.



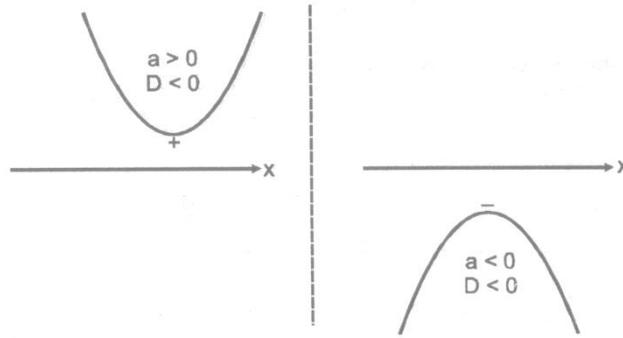
Roots are real & distinct

- (ii) If $D = 0$, the parabola will just touch the x-axis at one point and vice-versa.



Roots are equal

- (iii) If $D < 0$, the parabola will not intersect x-axis at all and vice-versa.



Roots are imaginary

Remarks

- ★ $\forall x \in \mathbb{R}, y > 0$ only if $a > 0$ & $D \equiv b^2 - 4ac < 0$
- ★ $\forall x \in \mathbb{R}, y < 0$ only if $a < 0$ & $D \equiv b^2 - 4ac < 0$

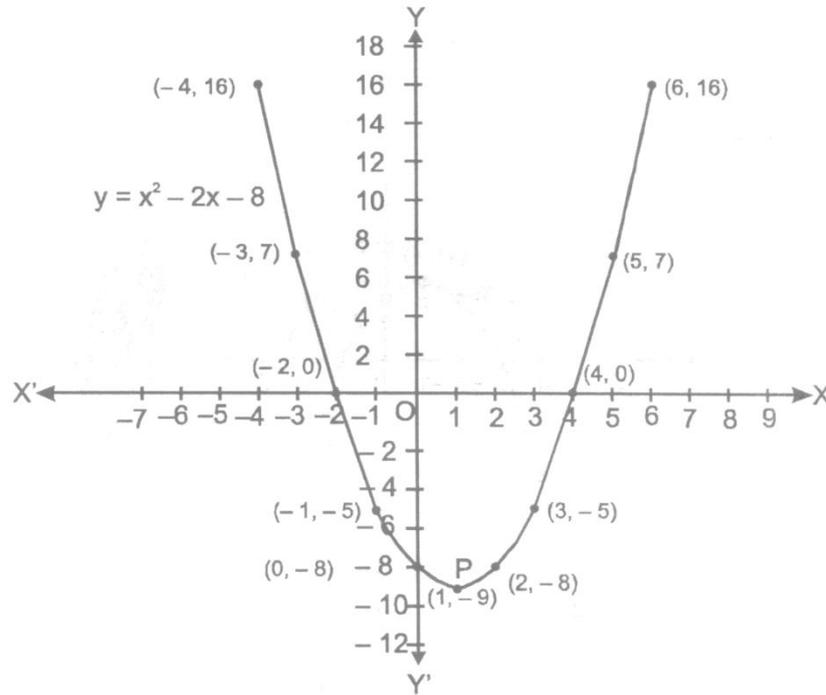
Examples: Draw the graph of the polynomial $f(x) = x^2 - 2x - 8$

Sol. Let $y = x^2 - 2x - 8$.

The following table gives the values of y or $f(x)$ for various values of x .

x	-4	-3	-2	-1	0	1	2	3	4	5	6
$y = x^2 - 2x - 8$	16	7	0	-5	-8	-9	-8	-5	0	7	16

Let us plot the points $(-4, 16)$, $(-3, 7)$, $(-2, 0)$, $(-1, -5)$, $(0, -8)$, $(1, -9)$, $(2, -8)$, $(3, -5)$, $(4, 0)$, $(5, 7)$ and $(6, 16)$ on a graphs paper and draw a smooth free hand curve passing through these points. The curve thus obtained represents the graphs of the polynomial $f(x) = x^2 - 2x - 8$. This is called a parabola. The lowest point P, called a minimum points, is the vertex of the parabola. Vertical line passing through P is called the axis of the parabola. Parabola is symmetric about the axis. So, it is also called the line of symmetry.



Observations:

From the graphs of the polynomial $f(x) = x^2 - 2x - 8$, following observations can be drawn :

- (i) The coefficient of x^2 in $f(x) = x^2 - 2x - 8$ is 1 (a positive real number) and so the parabola opens upwards.
- (ii) $D = b^2 - 4ac = 4 + 32 = 36 > 0$. So, the parabola cuts X-axis at two distinct points.
- (iii) On comparing the polynomial $x^2 - 2x - 8$ with $ax^2 + bx + c$, we get $a = 1$, $b = -2$ and $c = -8$.

The vertex of the parabola has coordinates $(1, -9)$ i.e. $\left(\frac{-b}{2a}, \frac{-D}{4a}\right)$, where $D \equiv b^2 - 4ac$.

(iv) The polynomial $f(x) = x^2 - 2x - 8 = (x - 4)(x + 2)$ is factorizable into two distinct linear factors $(x - 4)$ and $(x + 2)$. So, the parabola cuts X-axis at two distinct points $(4, 0)$ and $(-2, 0)$. the x-coordinates of these points are zeros of $f(x)$.

Example: Draw the graphs of the quadratic polynomial $f(x) = 3 - 2x - x^2$.

Sol. Let $y = f(x)$ or, $y = 3 - 2x - x^2$.

Let us list a few values of $y = 3 - 2x - x^2$ corresponding to a few values of x as follows :

x	-5	-4	-3	-2	-1	0	1	2	3	4
$y = 3 - 2x - x^2$	-12	-5	0	3	4	3	0	-5	-12	-21

Thus, the following points lie on the graph of the polynomial $y = 3 - 2x - x^2$:

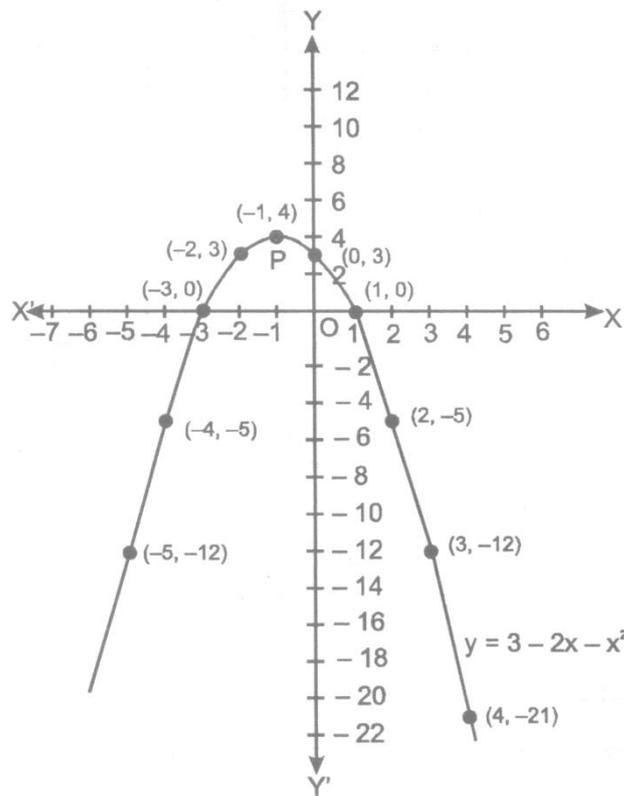
$(-5, -12), (-4, -5), (-3, 0), (-2, 3), (-1, 4), (0, 3), (1, 0), (2, -5), (3, -12)$ and $(4, -21)$.

Let plot these points on a graph paper and draw a smooth free hand curve passing through these points to obtain the graphs of $y = 3 - 2x - x^2$. The curve thus obtained represents a parabola, as shown in figure. The highest point $P(-1, 4)$,



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is called a maximum points, is the vertex of the parabola. Vertical line through P is the axis of the parabola. Clearly, parabola is symmetric about the axis.



Observations:

Following observations from the graph of the polynomial $f(x) = 3 - 2x - x^2$ is as follows :

- The coefficient of x^2 in $f(x) = 3 - 2x - x^2$ is -1 i.e. a negative real number and so the parabola opens downwards.
- $D \equiv b^2 - 4ac = 4 + 12 = 16 > 0$. So, the parabola cuts x -axis two distinct points.
- On comparing the polynomial $3 - 2x - x^2$ with $ax^2 + bx + c$, we have $a = -1$, $b = -2$ and $c = 3$. The vertex of the parabola is at the point $(-1, 4)$ i.e. at $\left(\frac{-b}{2a}, \frac{-D}{4a}\right)$, where $D = b^2 - 4ac$.
- The polynomial $f(x) = 3 - 2x - x^2 = (1 - x)(x + 3)$ is factorizable into two distinct linear factors $(1 - x)$ and $(x + 3)$. So, the parabola cuts X -axis at two distinct points $(1, 0)$ and $(-3, 0)$. The co-ordinates of these points are zeros of $f(x)$.

Graph of A Cubic Polynomial:

Graphs of a cubic polynomial does not have a fixed standard shape. Cubic polynomial graphs will always cross X -axis at least once and at most thrice.

Examples: Draw the graphs of the polynomial $f(x) = x^3 - 4x$.

Sol. Let $y = f(x)$ or, $y = x^3 - 4x$.

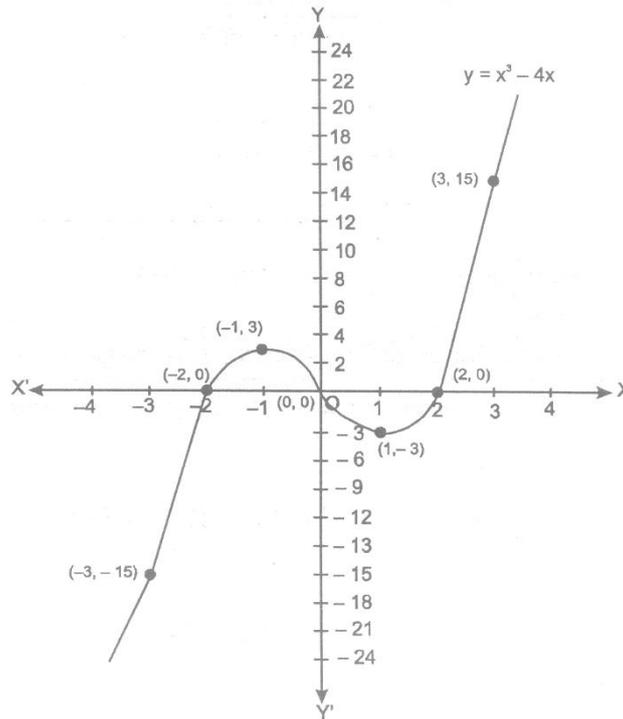
The values of y for variable value of x are listed in the following table :



x	-3	-2	-1	0	1	2	3
$y = x^3 - 4x$	-15	0	3	0	-3	0	15

Thus, the curve $y = x^3 - 4x$ passes through the points $(-3, -15)$, $(-2, 0)$, $(-1, 3)$, $(0, 0)$, $(1, -3)$, $(2, 0)$, $(3, 15)$, $(4, 48)$ etc.

Plotting these points on a graph paper and drawing a free hand smooth curve through these points, we obtain the graph of the given polynomial as shown figure.



Observations:

For the graphs of the polynomial $f(x) = x^3 - 4x$, following observations are as follows :-

- (i) The polynomial $f(x) = x^3 - 4x = x(x^2 - 4) = x(x - 2)(x + 2)$ is factorizable into three distinct linear factors. The curve $y = f(x)$ also cuts X-axis at three distinct points.
- (ii) We have, $f(x) = x(x - 2)(x + 2)$

Therefore 0, 2 and -2 are three zeros of $f(x)$. The curve $y = f(x)$ cuts X-axis at three points O (0, 0), P(2, 0) and Q (-2, 0).

Relationship Between Zeros and Coefficients of A Quadratic Polynomial:

Let α and β be the zeros of a quadratic polynomial $f(x) = ax^2 + bx + c$. By factor theorem $(x - \alpha)$ and $(x - \beta)$ are the factors of $f(x)$.

$\therefore f(x) = k(x - \alpha)(x - \beta)$ are the factors of $f(x)$

$\Rightarrow ax^2 + bx + c = k\{x^2 - (\alpha + \beta)x + \alpha\beta\}$

$\Rightarrow ax^2 + bx + c = kx^2 - k(\alpha + \beta)x + k\alpha\beta$



Comparing the coefficients of x^2 , x and constant terms on both sides, we get

$$a = k, b = -k(\alpha + \beta) \text{ and } k\alpha\beta$$

$$\Rightarrow \alpha + \beta = -\frac{b}{a} \text{ and } \alpha\beta = \frac{c}{a}$$

$$\Rightarrow \alpha + \beta = -\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2} \text{ and } \alpha\beta = \frac{\text{Constant term}}{\text{Coefficient of } x^2}$$

Hence,

$$\text{Sum of the zeros} = -\frac{b}{a} = -\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2}$$

$$\text{Product of the zeros} = \frac{c}{a} = \frac{\text{Constant term}}{\text{Coefficient of } x^2}$$

Remarks:

If α and β are the zeros of a quadratic polynomial $f(x)$. Then, the polynomial $f(x)$ is given by

$$f(x) = k\{x^2 - (\alpha + \beta)x + \alpha\beta\}$$

$$\text{or } f(x) = k\{x^2 - (\text{Sum of the zeros})x + \text{Product of the zeros}\}$$

Examples: Find the zeros of the quadratic polynomial $f(x) = x^2 - 2x - 8$ and verify the relationship between the zeros and their coefficients.

Sol. $f(x) = x^2 - 2x - 8$

$$\Rightarrow f(x) = x^2 - 4x + 2x - 8 \Rightarrow f(x) = x(x - 4) + 2(x - 4)]$$

$$\Rightarrow f(x) = (x - 4)(x + 2)$$

Zeros of $f(x)$ are given by $f(x) = 0$

$$\Rightarrow x^2 - 2x - 8 = 0 \Rightarrow (x - 4)(x + 2) = 0$$

$$\Rightarrow x = 4 \text{ or } x = -2$$

So, $\alpha = 4$ and $\beta = -2$

\therefore sum of zeros $\alpha + \beta$

$$= 4 - 2 = 2$$

Also, sum of zeros $= -\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2} = \frac{-(-2)}{1} = 2$

So, sum of zeros $= \alpha + \beta = -\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2}$

Now, product of zeros $= \alpha\beta$

$$= (4)(-2) = -8$$



$$\text{Also, product of zeros} = \frac{\text{Constant term}}{\text{Coefficient of } x^2} = \frac{-8}{1} = -8$$

$$\therefore \text{Product of zeros} = \frac{\text{Constant term}}{\text{Coefficient of } x^2} = \alpha\beta.$$

Examples: Find a quadratic polynomial whose zeros are $5 + \sqrt{2}$ and $5 - \sqrt{2}$

Sol. Given $\alpha = 5 + \sqrt{2}, \beta = 5 - \sqrt{2}$

$$\therefore f(x) = k\{x^2 - x(\alpha + \beta) + \alpha\beta\}$$

$$\text{Here, } \alpha + \beta = 5 + \sqrt{2} + 5 - \sqrt{2} = 10$$

$$\text{and } \alpha\beta = (5 + \sqrt{2})(5 - \sqrt{2})$$

$$= 25 - 2 = 23$$

$$\therefore f(x) = k\{x^2 - 10x + 23\}, \text{ where, } k \text{ is any non-zero real number.}$$

Examples: Sum of product of zeros of quadratic polynomial are 5 and 17 respectively. Find the polynomial.

Sol. Given : Sum of zeros = 5 and product of zeros = 17

So, quadratic polynomial is given by

$$\Rightarrow f(x) = k\{x^2 - x(\text{sum of zeros}) + \text{product of zeros}\}$$

$$\Rightarrow f(x) = k\{x^2 - 5x + 17\}, \text{ where, } k \text{ is any non-zero real number,}$$

Relationship Between Zeros and Coefficients of A Cubic Polynomial:

Let α, β, γ be the zeros of a cubic polynomial $f(x) = ax^3 + bx^2 + cx + d$, $a \neq 0$. Then, by factor theorem, $x - \alpha, x - \beta$ and $x - \gamma$ are factors of $f(x)$. Also, $f(x)$ being a cubic polynomial cannot have more than three linear factors.

$$\therefore f(x) = k(x - \alpha)(x - \beta)(x - \gamma)$$

$$\Rightarrow ax^3 + bx^2 + cx + d = k(x - \alpha)(x - \beta)(x - \gamma)$$

$$\Rightarrow ax^3 + bx^2 + cx + d = k\{x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)x - \alpha\beta\gamma\}$$

$$\Rightarrow ax^3 + bx^2 + cx + d = kx^3 - k(\alpha + \beta + \gamma)x^2 + k(\alpha\beta + \beta\gamma + \gamma\alpha)x - k\alpha\beta\gamma$$

Comparing the coefficients of x^3, x^2, x and constant terms on both sides, we get

$$a = k, b = -k(\alpha + \beta + \gamma), c = k(\alpha\beta + \beta\gamma + \gamma\alpha) \text{ and } d = -k\alpha\beta\gamma$$

$$\Rightarrow \alpha + \beta + \gamma = -\frac{b}{a}$$



$$\Rightarrow \alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$$

And, $\alpha\beta\gamma = -\frac{d}{a}$

$$\Rightarrow \text{Sum of the zeros} = -\frac{b}{a} = -\frac{\text{Coefficient of } x^2}{\text{Coefficient of } x^3}$$

$$\Rightarrow \text{Sum of the products of the zeros taken two at a time} = \frac{c}{a} = \frac{\text{Coefficient of } x}{\text{Coefficient of } x^3}$$

$$\Rightarrow \text{Product of the zeros} = -\frac{d}{a} = -\frac{\text{Constant term}}{\text{Coefficient of } x^3}$$

Remarks:

Cubic polynomial having α, β and γ as its zeros is given by

$$f(x) = k(x - \alpha)(x - \beta)(x - \gamma)$$

$$f(x) = k \{x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)x - \alpha\beta\gamma\} \text{ where } k \text{ is any non-zero real number.}$$

Example: Verify that $\frac{1}{2}, 1 - 2$ are zeros of cubic polynomial $2x^3 + x^2 - 5x + 2$. Also verify the relationship between, the zeros and their coefficients.

Sol. $f(x) = 2x^3 + x^2 - 5x + 2$

$$f\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^2 - 5\left(\frac{1}{2}\right) + 2$$

$$= \frac{1}{4} + \frac{1}{4} - \frac{5}{2} + 2$$

$$f(1) = 2(1)^3 + (1)^2 - 5(1) + 2 = 2 + 1 - 5 + 2 = 0.$$

$$f(-2) = 2(-2)^3 + (-2)^2 - 5(-2) + 2 = -16 + 4 + 10 + 2 = 0.$$

Let $\alpha = \frac{1}{2}, \beta = 1$ and $\gamma = -2$

Now, Sum of zeros $= \alpha + \beta + \gamma$

$$= \frac{1}{2} + 1 - 2$$

$$= -\frac{1}{2}$$



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$$\begin{aligned}\text{Also, sum of zeros} &= -\frac{(\text{Coefficient of } x^2)}{\text{Coefficient of } x^3} \\ &= -\frac{1}{2}\end{aligned}$$

$$\text{So, sum of zeros } = \alpha + \beta + \gamma = -\frac{(\text{Coefficient of } x^2)}{\text{Coefficient of } x^3}$$

$$\text{Sum of product of zeros taken two at a time } = \alpha\beta + \beta\gamma + \gamma\alpha$$

$$\begin{aligned}&= \frac{1}{2} \times 1 + 1 \times (-2) + (-2) \times \frac{1}{2} \\ &= -\frac{5}{2}\end{aligned}$$

$$\text{Also, } \alpha\beta + \beta\gamma + \gamma\alpha = \frac{\text{Coefficient of } x}{\text{Coefficient of } x^3} = \frac{-5}{2}$$

$$\text{So, sum of product of zeros taken two at a time } = \alpha\beta + \beta\gamma + \gamma\alpha = \frac{\text{Coefficient of } x}{\text{Coefficient of } x^3}$$

$$\text{Now, Product of zeros } = \alpha\beta\gamma$$

$$= \left(\frac{1}{2}\right)(1)(-2) = -1$$

$$\text{Also, product of zeros } = \frac{\text{Constant term}}{\text{Coefficient of } x^3}$$

$$= \frac{-2}{2} = -1$$

$$\therefore \text{Product zeros } = \alpha\beta\gamma = -\frac{\text{Constant term}}{\text{Coefficient of } x^3}$$

Example: Find a polynomial with the sum, sum of the product of its zeros taken two at a time, and product its zeros as 3, -1 and -3 respectively.

Sol. Given $\alpha + \beta + \gamma = 3$, $\alpha\beta + \beta\gamma + \gamma\alpha = -1$ and $\alpha\beta\gamma = -3$

$$\text{So, polynomial } f(x) = k \{x^3 - x^2(\alpha + \beta + \gamma) + x(\alpha\beta + \beta\gamma + \gamma\alpha) - \alpha\beta\gamma\}$$

$$f(x) = k \{x^3 - 3x^2 - x + 3\}, \text{ where } k \text{ is any non-zero real number.}$$

Value of a Polynomial:



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The value of a polynomial $f(x)$ at $x = \alpha$ is obtained by substituting $x = \alpha$ in the given polynomial and is denoted by $f(\alpha)$.

For example : If $f(x) = 2x^3 - 13x^2 + 17x + 12$ then its value at $x = 1$ is

$$f(1) = 2(1)^3 - 13(1)^2 + 17(1) + 12 = 2 - 13 + 17 + 12 = 18.$$

Zeros of roots of a polynomial:

A real number ' a ' is a zero of a polynomial $f(x)$, if $f(a) = 0$, Here ' a ' is called a root of the equation $f(x) = 0$.

Example: Show that $x = 2$ is a root of $2x^3 + x^2 - 7x - 6$

Sol. $p(x) = 2x^3 + x^2 - 7x - 6$.

Then, $p(2) = 2(2)^3 + (2)^2 - 7(2) - 6 = 16 + 4 - 14 - 6 = 0$

Hence $x = 2$ is a root of $p(x)$.

Example: If $x = \frac{4}{3}$ is a root of the polynomial $f(x) = 6x^3 - 11x^2 + kx - 20$ then find the value of k .

Sol. $f(x) = 6x^3 - 11x^2 + kx - 20$

$$f\left(\frac{4}{3}\right) = 6\left(\frac{4}{3}\right)^3 - 11\left(\frac{4}{3}\right)^2 + k\left(\frac{4}{3}\right) - 20 = 0$$

$$\Rightarrow 6\left(\frac{64}{27}\right) - 11\left(\frac{16}{9}\right) + \frac{4k}{3} - 20 = 0 \Rightarrow 6\left(\frac{64}{27}\right) - 11\left(\frac{16}{9}\right) + \frac{4k}{3} - 20 = 0$$

$$\Rightarrow 128 - 176 + 12k - 180 = 0$$

$$\Rightarrow 12k + 128 - 356 = 0$$

$$\Rightarrow 12k = 228$$

$$\Rightarrow k = 19.$$

Example: If $x = 2$ & $x = 0$ are roots of the polynomials $f(x) = 2x^3 - 5x^2 + ax + b$, then find the values of a and b .

Sol. $f(2) = 2(2)^3 - 5(2)^2 + a(2) + b = 0$

$$\Rightarrow 16 - 20 + 2a + b = 0 \Rightarrow 2a + b = 4 \quad \dots(i)$$

$$\Rightarrow f(0) = 2(0)^3 - 5(0)^2 + a(0) + b = 0 \Rightarrow b = 0$$

$$\Rightarrow 2a = 4 \Rightarrow a = 2, b = 0.$$



Factor Theorem:

Let $p(x)$ be a polynomial of degree greater than or equal to 1 and 'a' be a real number such that $p(a) = 0$. then $(x - a)$ is a factor of $p(x)$. Conversely, if $(x - a)$ is a factor of $p(x)$, then $p(a) = 0$.

Example: Show that $x + 1$ and $2x - 3$ are factors of $2x^3 - 9x^2 + x + 12$.

Sol. To prove that $(x + 1)$ and $(2x - 3)$ are factors of $p(x) = 2x^3 - 9x^2 + x + 12$ it is sufficient to show that $p(-1)$ and $p\left(\frac{3}{2}\right)$ both are equal to zero.

$$p(-1) = 2(-1)^3 - 9(-1)^2 + (-1) + 12 = -2 - 9 - 1 + 12 = -12 + 12 = 0$$

$$\begin{aligned} \text{And } p\left(\frac{3}{2}\right) &= 2\left(\frac{3}{2}\right)^3 - 9\left(\frac{3}{2}\right)^2 + \left(\frac{3}{2}\right) + 12 \\ &= \frac{27}{4} - \frac{81}{4} + \frac{3}{2} + 12 = \frac{27 - 81 + 6 + 48}{4} = \frac{-81 + 81}{4} = 0 \end{aligned}$$

Example: Find α and β if $x + 1$ and $x + 2$ are factors of $p(x) = x^3 + 3x^2 - 2\alpha x + \beta$.

Sol. $x + 1$ and $x + 2$ are the factor of $p(x)$.

$$\text{Then, } p(-1) = 0 \text{ \& } p(-2) = 0$$

$$\text{Therefore, } p(-1) = (-1)^3 + 3(-1)^2 - 2\alpha(-1) + \beta = 0$$

$$\Rightarrow -1 + 3 + 2\alpha + \beta = 0 \Rightarrow \beta = -2\alpha - 2 \quad \dots(i)$$

$$p(-2) = (-2)^3 + 3(-2)^2 - 2\alpha(-2) + \beta = 0$$

$$\Rightarrow -8 + 12 + 4\alpha + \beta = 0 \Rightarrow \beta = -4\alpha - 4 \quad \dots(ii)$$

From equation (1) and (2)

$$-2\alpha - 2 = -4\alpha - 4$$

$$\Rightarrow 2\alpha = -2 \Rightarrow \alpha = -1$$

$$\text{Put } \alpha = -1 \text{ equation (1)} \Rightarrow \beta = -2(-1) - 2 = 2 - 2 = 0.$$

Hence $\alpha = -1, \beta = 0$

Example: What must be added to $3x^3 + x^2 - 22x + 9$ so that the result is exactly divisible by $3x^2 + 7x - 6$.

Sol. Let $p(x) = 3x^3 + x^2 - 22x + 9$ and $q(x) = 3x^2 + 7x - 6$

We know if $p(x)$ is divided by $q(x)$ which is quadratic polynomial then the remainder be $r(x)$ and degree of $r(x)$ is less than $q(x)$ or Divisor.

\therefore By long division method



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Let we added $ax + b$ (linear polynomial) in $p(x)$, so that $p(x) + ax + b$ is exactly divisible by $3x^2 + 7x - 6$.

Hence, $p(x) + ax + b = s(x) = 3x^3 - x^2 - 22x + 9 + ax + b = 3x^3 + x^2 x(22 - a) + (9 + b)$.

$$\begin{array}{r}
 \overline{) 3x^3 + x^2 - x(22 - a) + 9 + b} \\
 \underline{- 3x^3 \pm 7x^2 + - 6x} \\
 - 6x^2 + 6x - (22 - a)x + 9 + b \\
 - 6x^2 x(-16 + a) + 9 + b \\
 \underline{+ -6x^2 + -14x \pm 12} \\
 x(-2 + a) + (b - 3) = 0
 \end{array}$$

Hence, $x(a - 2) + b - 3 = 0$. $x + 0$

$$\Rightarrow a - 2 = 0 \text{ \& } b - 3 = 0$$

$$\Rightarrow a = 2 \text{ and } b = 3$$

Hence if in $p(x)$ we added $2x + 3$ then it is exactly divisible by $3x^2 + 7x - 6$.

Example: What must be subtracted from $x^3 - 6x^2 - 15x + 80$ so that the result is exactly divisible by $x^2 + x - 12$.

Sol. Let $ax + b$ be subtracted from $p(x) = x^3 - 6x^2 - 15x + 80$ so that it is exactly divisible by $x^2 + x - 12$.

$$\therefore s(x) = x^3 - 6x^2 - 15x + 80 - (ax + b)$$

$$= x^3 - 6x^2 - (15 + a)x + (80 - b)$$

Dividend = Divisor \times quotient + remainder

But remainder will be zero.

$$\therefore \text{Dividend} = \text{Divisor} \times \text{quotient}$$

$$\Rightarrow s(x) = (x^2 + x - 12) \times \text{quotient}$$

$$\Rightarrow s(x) = x^3 - 6x^2 - (15 + a)x + (80 - b)$$

$$\begin{array}{r}
 \overline{) x^3 - 6x^2 - x(15 + a) + 80 - b} \\
 \underline{- x^3 \pm x^2 \mp 12x} \\
 - 7x^2 + 12x - (15 + a)x + 80 - b \\
 - 7x^2 + (-3 - a)x + 80 - b \\
 \underline{\mp 7x^2 \mp 7x \quad \pm 84} \\
 x(4 - a) + (-4 - b) = 0
 \end{array}$$

Hence, $x(4 - a) + (-4 - b) = 0$. $x + 0$

$$\Rightarrow 4 - a = 0 \text{ \& } (-4 - b) = 0$$



$$\Rightarrow a = 4 \text{ and } b = -4$$

Hence, if in $p(x)$ we subtract $4x - 4$ then it is exactly divisible by $x^2 + x - 12$.

Example: Using factor theorem, factorize : $p(x) = 2x^4 - 7x^3 - 13x^2 + 63x - 45$.

Sol. $45 \Rightarrow \pm 1, \pm 3, \pm 5, \pm 9, \pm 15, \pm 45$

If we put $x = 1$ in $p(x)$

$$p(1) = 2(1)^4 - 7(1)^3 - 13(1)^2 + 63(1) - 45$$

$$p(1) = 2 - 7 - 13 + 63 - 45 = 65 - 65 = 0$$

$\therefore x = 1$ or $x - 1$ is a factor of $p(x)$.

Similarly if we put $x = 3$ in $p(x)$

$$p(3) = 2(3)^4 - 7(3)^3 - 13(3)^2 + 63(3) - 45$$

$$p(3) = 162 - 189 - 117 + 189 - 45 = 162 - 162 = 0$$

Hence, $x = 3$ or $(x - 3) = 0$ is the factor of $p(x)$.

$$p(x) = 2x^4 - 7x^3 - 13x^2 + 63x - 45$$

$$\therefore p(x) = 2x^3(x - 1) - 5x^2(x - 1) - 18x(x - 1) + 45(x - 1)$$

$$\Rightarrow p(x) = (x - 1)(2x^3 - 5x^2 - 18x + 45)$$

$$\Rightarrow p(x) = (x - 1)(2x^3 - 5x^2 - 18x + 45)$$

$$\Rightarrow p(x) = (x - 1)[2x^2 - (x - 3) + x(x - 3) - 15(x - 3)]$$

$$\Rightarrow p(x) = (x - 1)(x - 3)(2x^2 + x - 15)$$

$$\Rightarrow p(x) = (x - 1)(x - 3)(2x^2 + 6x - 5x - 15)$$

$$\Rightarrow p(x) = (x - 1)(x - 3)[2x(x + 3) - 5(x + 3)]$$

$$\Rightarrow p(x) = (x - 1)(x - 3)(x + 3)(2x - 5).$$

4.10 REMAINDER THEOREM :

Let $p(x)$ be any polynomial of degree greater than or equal to one and 'a' be any real number. If $p(x)$ is divided by $x - a$, then the remainder is equal to $p(a)$.

Let $q(x)$ be the quotient and $r(x)$ be the remainder when $p(x)$ is divided by $(x - a)$, then

$$\text{Dividend} = \text{Divisor} \times \text{Quotient} + \text{Remainder}$$

Example: Find the remainder when $f(x) = x^3 - 6x^2 + 2x - 4$ is divided by $g(x) = 1 - 2x$.

Sol. $1 - 2x = 0 \Rightarrow 2x = 1 \Rightarrow x = \frac{1}{2}$



$$f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^3 - 6\left(\frac{1}{2}\right)^2 + 2\left(\frac{1}{2}\right) - 4 = \frac{1}{8} - \frac{3}{2} + 1 - 4 = \frac{1 - 12 + 8 - 32}{8} = -\frac{35}{8}$$

Example: Apply division algorithm to find the quotient $q(x)$ and remainder $r(x)$ on dividing $f(x) = 10x^4 + 17x^3 - 62x^2 + 30x - 3$ by $b(x) = 2x^2 - x + 1$.

Sol.

$$\begin{array}{r} 5x^2 + 11x - 28 \\ 2x^2 - x + 1 \overline{) 10x^4 + 17x^3 - 62x^2 + 30x - 3} \\ \underline{-10x^4 + -5x^3 + 5x^2} \\ 22x^3 - 67x^2 + 30x - 3 \\ \underline{-22x^3 + -11x^2 + 11x} \\ -56x^2 + 19x - 3 \\ \underline{+56x^2 + 28x - 28} \\ -9x + 25 \end{array}$$

So, quotient $q(x) = 5x^2 + 11x - 28$ and remainder $r(x) = -9x + 25$.

Now, dividend = Quotient \times Divisor + Remainder

$$= (5x^2 + 11x - 28)(2x^2 - x + 1) + (-9x + 25)$$

$$= 10x^4 - 5x^3 + 5x^2 + 22x^3 - 11x^2 + 11x - 56x^2 + 28x - 28 - 9x + 25$$

$$= 10x^4 + 17x^3 - 62x^2 + 30x - 3$$

Hence, the division algorithm is verified.

Example: Find all the zeros of the polynomial $f(x) = 2x^4 - 2x^3 - 7x^2 + 3x + 6$, if two of its zeros are $-\sqrt{\frac{3}{2}}$ and $\sqrt{\frac{3}{2}}$.

Sol. Since $-\sqrt{\frac{3}{2}}$ and $\sqrt{\frac{3}{2}}$ are zeros of $f(x)$.

Therefore, $\left(x + \sqrt{\frac{3}{2}}\right)\left(x - \sqrt{\frac{3}{2}}\right) = \left(x^2 - \frac{3}{2}\right) = \frac{2x^2 - 3}{2}$ or $2x^2 - 3$ is a factor of $f(x)$.

$$\begin{array}{r} x^2 - x - 2 \\ 2x^2 - 3 \overline{) 2x^4 - 2x^3 - 7x^2 + 3x + 6} \\ \underline{-2x^4} \\ -2x^3 - 4x^2 + 3x + 6 \\ \underline{+ 2x^3} \\ -4x^2 + 6 \\ \underline{-4x^2 + 6} \\ + \\ 0 \end{array}$$

$$\therefore 2x^4 - 2x^3 - 7x^2 + 3x + 6 = (2x^2 - 3)(x^2 - x - 2)$$

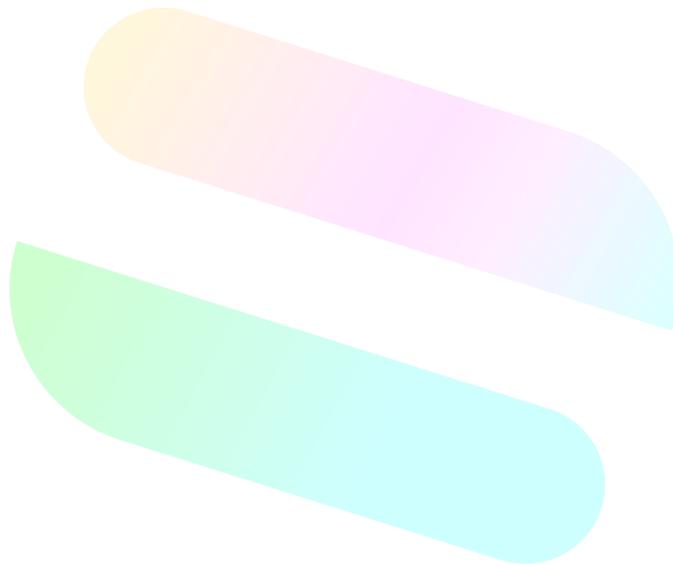


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$$= (2x^2 - 3)(x - 2)(x + 1)$$

$$= 2\left(x + \sqrt{\frac{3}{2}}\right)\left(x - \sqrt{\frac{3}{2}}\right)(x - 2)(x + 1)$$

So, the zeros are $-\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}}, 2, -1$



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