SHIKSHA NATION

Real Number

Divisibility:

A non-zero integer 'a' is said to divide an integer 'b' if there exists an integer 'c' such that b = ac. The integer 'b' is called dividend, integer 'a' is known as the divisor and integer 'c' is known as the quotient.

For example, 5 divides 35 because there is an integer 7 such that $35 = 5 \times 7$.

If a non-zero integer 'a' divides an integer b, then it is written as a | b and read as 'a a divides b', a/b is written to indicate that b is not divisible by a.

Euclid's division lemma:

Let 'a' and 'b' be any two positive integers. Then, there exists unique integers 'q' and 'r' such that a = b + r, where 0 r b. If b|a, than r = 0.

Example: Show that any positive odd integer is of the form 6q + 1 or, 6q + 3 or, 6q + 5, where q is some integer.

Sol. Let 'a' be any positive integer and b = 6. Then, by Euclid's division lemma there exists integers 'a' and 'r' such that

a = 6q + r, where 0 r < 6.

$\Rightarrow$ a = 6q or, a = 6q + 1 or, a = 6q + 2 or, a = 6a + 3 or, a = 6q + 4 or, a = 6q + 5.

[$\therefore$ 0 r < 6 $\therefore$ r = 0, 1,2,3,4,5]

$\Rightarrow$ a = 6q + 1 or, a = 6q + 3 or, a = 6q + 5.

[$\therefore$ a is an odd integer, $\therefore$ $\therefore$ 6q, a $\neq$ 6q + 2, a $\neq$ 6q + 4]

Hence, any odd integer is of the form 6q + 1 or, 6q + 3 or, 6q + 5.

Example: Use Euclid's Division Lemma to show that the cube of any positive integer is of the form 9m, 9m + 1 or 9 m + 8, for some integer q.

Sol. Let x be any positive integer. Then, it is of the form 3q or, 3q + 1 or, 3 + 2.

Case - I When x = 3q

$\Rightarrow$ $x^3 = (3q)^3 = 27q^3 = 9(3q^3) = 9m$, where $m = 9q^3$

Case - II when x = 3q + 1

$\Rightarrow$ $x^3 = (3q + 1)^3$

$\Rightarrow$ $x^3 = 2q^3 + 27q^2 + 9q + 1$

$\Rightarrow$ $x^3 = 9q(3q^2 + 3q + 1) + 1$

$\Rightarrow$ $x^3 = 9m + 1$, where $m = q(3q^2 + 3q + 1)$.

Case -III when x = 3q + 2

$\Rightarrow \quad x^3 = (3q + 2)^3$

$\Rightarrow \quad x^3 = 27q^3 + 54q^2 + 36q + 8$

$\Rightarrow \quad x^3 = 9q(3q^2 + 6q + 4) + 8$

$\Rightarrow \quad x^3 = 9m + 8$, where $m = 3q^2 + 6q + 4)$

Hence, x^3 is either of the form 9m of 9m + 1 or 9m + 8.

Example: Prove that the square of any positive integer of the form 5q + 1 is of the same form.

Sol. Let x be any positive's integer of the form 5q + 1.

When x= 5q + 1

$x^2 = 25q^2 + 10q + 1$

$x^2 = 5(5q + 2) + 1$

Let m = q (5q + 2).

$x^2 = 5m + 1.$

Hence, x^2 is of the same form i.e. 5m + 1.

Euclid's division algorithm:

If 'a' and 'b' are positive integers such that a = bq + r, then every common divisor of 'a' and 'b' is a common divisor of 'b' and 'r' and vice-versa.

Example: Use Euclid's division algorithm to find the H.C.F. of 196 and 38318.

Sol. Applying Euclid's division lemma to 196 and 38318.

$38318 = 195 \times 196 + 98$

$196 = 98 \times 2 + 0$

The remainder at the second stage is zero. So, the H.C.F. of 38318 and 196 is 98.

Example: If the H.C.F. of 657 and 963 is expressible in the form 657x + 963 × (-15), find x.

Sol. Applying Euclid's division lemma on 657 and 963.

$963 = 657 \times 1 + 306$

$657 = 306 \times 2 + 45$

$306 = 45 \times 6 + 36$

$45 = 36 \times 1 + 9$

$36 = 9 \times 4 + 0$

So, the H.C.F. of 657 and 963 is 9.

Given : $657x + 963 \times (-15)$ = H.C.F. of 657 and 963.

$657x + 963 \times (-15) = 9$

$657x \quad = 9 + 963 \times 15$

$657x \quad = 14454$

$$x = \frac{14454}{657} = 22\ b$$

Example: What is the largest number that divides 626, 3127 and 15628 and leaves remainders of 1, 2 and 3 respectively.

Sol. Clearly, the required number is the H.C.F. of the number 626 - 1 = 625, 3127 - 2 3125 and 15628 - 3 = 15625.

15628 - 3 = 15625.

Using Euclid's division lemma to find the H.C.F. of 625 and 3125.

$3125 = 625 \times 5 + 0$

Clearly, H.C.F. of 625 and 3125 is 625.

Now, H.C.F. of 625 and 15625

$15625 = 625 \times 25 + 0$

So, the H.C.F. of 625 and 15625 is 625.

Hence, H.C.F. of 625, 3125 and 15625 is 625.

Hence, the required number is 625.

Example: 144 cartons of coke cans and 90 cartons of Pepsi cans are to be stacked is a canteen. If each stack is of same height and is to contains cartons of the same drink, what would be the greatest number of cartons each stack would have ?

Sol. In order to arrange the cartons of the same drink is the same stack, we have to find the greatest number that divides 144 and 90 exactly. Using Euclid's algorithm, to find the H.C.F. of 144 and 90.

$144 = 90 \times 1 + 54$

$90 = 54 \times 1 + 36$

$54 = 36 \times 1 + 18$

$36 = 18 \times 2 + 0$

So, the H.C.F. of 144 and 90 is 18.

Number of cartons in each stack = 18.

Fundamental theorem of arithmetic:

Every composite number can be expressed as a product of primes, and this factorisation is unique, except for the order in which the prime factors occurs.

Some important results:

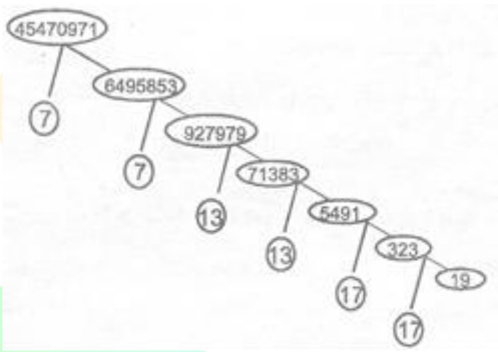
(i) Let ‘p’ be a prime number and ‘a’ be a positive integer. If ‘p’ divides a2, then ‘p’ divides ‘a’.

(ii) Let x be a rational number whose decimal expansion terminates. Then, x can be expressed in the form where p and q are co-primes, and prime factorisation of q is of the form 2m × 5n, where m, n are non-negative integers.

(iii) Let be a rational number, such that the prime factorisation of q is not of the form 2m × 5n where m, n are non - negative integers. Then, x has a decimal expansion which is non - terminating repeating.

Example: Determine the prime factors of 45470971.

Sol.



∴ $45470971 = 7^2 \times 13^2 \times 17^2 \times 19$.

Example: Check whether 6n can end with the digit 0 for any natural number.

Sol. Any positive integer ending with the digit zero is divisible by 5 and so its prime factorisations must contain the prime 5.

$6^n = (2 \times 3)^n = 2^n \times 3^n$

⇒ The prime in the factorisation of 6 n is 2 and 3.

⇒ 5 does not occur in the prime factorisation of 6n for any n.

⇒ 6 n does not end with the digit zero for any natural number n.

Example: Find the LCM and HCF of 84, 90 and 120 by applying the prime factorisation method.

Sol. $84 = 2^2 \times 3 \times 7$, $90 = 2 \times 3^2 \times$ and $120 = 2^3 \times 3 \times 5$.

Prime factors	Least exponent
2	1
3	1
5	0
7	0

$\therefore$ HCF $= 2^1 \times 3^1 = 6.$

Common prime factors	Greatest exponent
2	3
3	2
5	1
7	1

$\therefore$ LCM $= 2^3 \times 3^3 \times 5^1 \times 7^1$

$= 8 \times 9 \times 5 \times 7$

$= 2520.$

Example: In a morning walk three persons step off together, their steps measure 80 cm, 85 cm and 90 cm respectively. What is the minimum distance each should walk so that they can cover the distance in complete steps ?

Sol. Required minimum distance each should walk so, that they can cover the distance in complete step is the L.C.M. of 80 cm, 85 cm and 90 cm

$80 = 2^4 \times 5$

$85 = 5 + 17$

$90 = 2 \times 3^2 \times 5$

$\therefore$ LCM $= 2^4 \times 3^2 \times 5^1 \times 17^1$

LCM $= 16 \times 9 \times 5 \times 17$

LCM = 12240 cm, = 122 m 40 cm.

Example: Prove that $\sqrt{2}$ is an irrational number.

Sol. Let assume on the contrary that $\sqrt{2}$ is a rational number.

Then, there exists positive integer a and b such that $\sqrt{2} = \frac{a}{b}$ where, a and b are co primes i.e. their HCF is 1.

$\Rightarrow \quad \left(\sqrt{2}\right)^2 = \left(\frac{a}{b}\right)^2$

$\Rightarrow \quad 2 = \frac{a^2}{b^2}$

$\Rightarrow$ $a^2 = 2b^2$

$\Rightarrow$ a^2 is multiple of 2

a is a multiple of 2 ...(i)

$\Rightarrow$ a = 2c for some integer c.

$\Rightarrow$ $a^2 = 4c^2$

$\Rightarrow$ $2b^2 = 4c^2$

$\Rightarrow$ $b^2 = 2c^2$

$\Rightarrow$ b^2 is a multiple of 2

b is a multiple of 2 ...(ii)

From (i) and (ii), a and b have at least 2 as a common factor. But this contradicts the fact that a and b are co-prime. This means that is an irrational number.

Example: Prove that $3-\sqrt{5}$ is an irrational number.

Sol. Let assume that on the contrary that $3-\sqrt{5}$ is rational.

Then, there exist co-prime positive integers a and b such that,

$$3-\sqrt{5}=\frac{a}{b}$$

$$\Rightarrow \quad 3-\frac{a}{b}=\sqrt{5}$$

$$\Rightarrow \quad \frac{3b-a}{b}=\sqrt{5}$$

$\Rightarrow$ $\sqrt{5}$ is rational [$\because$ a,b, are integer $\therefore$ $\frac{3b-a}{b}$ is a rational number]

This contradicts the fact that $\sqrt{5}$ is irrational

Hence, $3-\sqrt{5}$ is an irrational number.

Examples: Without actually performing the long division, state whether $\frac{13}{3125}$ has terminating decimal expansion or not.

Sol. $$\frac{13}{3125}=\frac{13}{2^0\times 5^5}$$

This, shows that the prime factorisation of the denominator is of the form $2^m \times 5^n$.

Hence, it has terminating decimal expansion.

Example: What can you say about the prime factorisations of the denominators of the following rationals :

(i) 43.123456789 (ii) 43.

Sol. (i) Since, 43.123456789 has terminating decimal, so prime factorisations of the denominator is of the form $2^m \times 5^n$, where m, n are non - negative integers.

(ii) Since, 43. $43.\overline{123456789}$ has non-terminating repeating decimal expansion. So, its denominator has factors other than 2 or 5.

1. Can we have any n $\in$ N, where 7^n ends with the digit zero.

2. Without actually performing the long division, state whether the following rational number will have a terminating decimal expansion or non - terminating decimal expansion :

(i) $\frac{77}{210}$ (ii) $\frac{15}{1600}$

3 An army contingent of 616 members is to march behind and army band of 32 members in a parade. The two groups are to march in the same number of columns. What is the maximum number of columns in which they can march?

4. There is a circular path around a sports field. Sonia takes 18 minutes to drive one round of the field, while Ravi takes 12 minutes for the same. Suppose they both start at the same point and at the same time, and go in the same direction. After how many minutes will they meet again at the starting point ?

5. Write a rational number between $\sqrt{2}$ and $\sqrt{3}$. [CBSE - 2008]

6. Use Euclid's' Division Lemma to show that the square of any positive integer is either of the form 3m of 3m + 1 for some integer m. [CBSE - 2008]

7 Use Euclid's division algorithm to find the HCF of :

(i) 56 and 814 (ii) 6265 and 76254

8. Find the HCF and LCM of following using Fundamental Theorem of Arithmetic method.

(i) 426 and 576 (ii) 625, 1125 and 2125

9. Prove that $\sqrt{3}$ is an irrational number. [CBSE - 2008]

10. Prove that $\sqrt{5}$ is irrational number. [CBSE - 2008]

11. Prove that $5+\sqrt{2}$ is irrational.

12. Prove that $\sqrt{2}+\sqrt{3}$ is irrational.